THE ANALYSIS OF NONLINEAR SYSTEMS
SUBJECT TO RANDOM INPUTS

by

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INTRODUCTION

Modern, high performance control systems require accurate analysis of the effects of input noise and disturbances. Since noise and disturbances are generally random in nature, it is difficult to describe them with time functions. Nevertheless, such inputs can very often be represenced by other mathematical representations, such as statistical analysis. These techniques are not uncommon in linear system theory, and extension to nonlinear systems is the primary object of this report.

When a random input signal to a nonlinear control system can be described by certain types of statistical functions, a method of analysis may be developed for the nonlinear system. In the development of the method, the criterion for design is restricted to the minimization of the mean square or root mean square value of the error signal. The method involves replacement of the nonlinear element in the system by an equivalent device whose describing function characteristics are a functions of the rms signals at the input of nonlinear element. Two methods of deriving this statistical describing function are developed, one uses an analytical method to minimize the system error signal and the other requires the determination of the output probability distribution of the nonlinear device and defines the ratio of the rms value of the output distribution to the rms value of the input distribution as the statistical describing function. Both of these methods are developed for simple nonlinear devices in which the output is a function of the instantaneous value of the input, therefore, the analysis techniques of linear system theory can be made applicable with only slight modification.

One matter of particular interest, the analysis of jump phenomenon, is

also developed. An example of positioning servomechanism with a saturating amplifier that exhibits the jump phemomenon when the input is a white noise passing through the lag filter is presented.

A brief discussion of the fundamental statistical techniques is also given in order to provide the supporting background for the developement of the analysis procedures.

SOME FUNDAMENTAL CONCEPTS OF PROBABILITY THEORY

The following sections present a brief review of the different tools and techniques available when one is dealing with statistical random processes.

1. Probability density

A random time variable cannot be represented in the form of explicit equation which is valid over an extended period of time. However, the randomly varying quantity can be described by various statistical relationships. Basic among these is the probability function.

There are several ways to define the probability function. One of them is the relative frequency definition. In a random experiment, a random variable is defined as a single valued function ξ that associates a distinct number with each possible outcomes of the experiment. If an experiment is repeated n times and if the event that the random variable ξ assumes values between a and b occurs n_a times, then the probability of this event can be defined as

$$\text{prob.}\ (a < \xi < b) = \lim_{n \to \infty} \frac{n_a}{n} \tag{1}$$

In order to apply statistical methods conveniently to continuous functions, the probability that a particular event occurs is usually given in terms of a function called the probability density, $P_\xi(x)$,

$$\text{prob.}\ (a < \xi < b) = \int_a^b P_\xi(x)\, dx. \tag{2}$$

When the probability density is known, the mean value of the random variable $\bar{\xi}$ is

$$\bar{\xi} = \int_{-\infty}^{\infty} x p_{\xi}(x)\, dx \tag{3}$$

and the mean square value of the random variable $\overline{\xi^2}$ is

$$\overline{\xi^2} = \int_{-\infty}^{\infty} x^2 P_{\xi}(x)\, dx. \tag{4}$$

In the random time function $x(t)$, the random variable ξ represents the amplitude of the waveform at some time t. Then, Prob. $(x < \xi < x+dx)$ $= P_{\xi}(x)dx$ is the probability that the amplitude of a particular waveform falls in the interval $x < \xi < x+dx$ at time t. If the probability density of the amplitude of an ensemble is independent of time, the ensemble is said to be stationary. For the stationary random process, the ergodic hypothesis states that these ensemble averages are the same as the corresponding time averages, taken over an infinite period, of a single sample function. That is

$$\bar{\xi} = \int_{-\infty}^{\infty} x P_{\xi}(x)\, dx = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)\, dt \tag{5}$$

$$\overline{\xi^2} = \int_{-\infty}^{\infty} x^2 P_{\xi}(x)\, dx = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x^2(t)\, dt\ . \tag{6}$$

2. Normal Distribution Function

A general mathematical expression for a normal probability density is

$$P_{\xi}(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[- \frac{(x - \overline{\xi})^2}{2\sigma^2} \right] \tag{7}$$

where ξ is the standard deviation which is defined as the square root of variance $\overline{(\xi-\overline{\xi})^2}$ and $\overline{\xi}$ is the mean value as given in equation (2).

If $\overline{\xi}$ equal zero, equation (7) changes into the form

$$P_{\xi}(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[- \frac{x^2}{2\sigma^2} \right] \tag{8}$$

A graph of equation (8) is shown in Figure 1. The integral in equation (2) corresponds to the shaded area. Since the normal probability density is completely described by the standard deviation and the mean, it is a very convenient function with which to work. Many random quantities encountered in control problems have probability densities that may be considered normal densities.

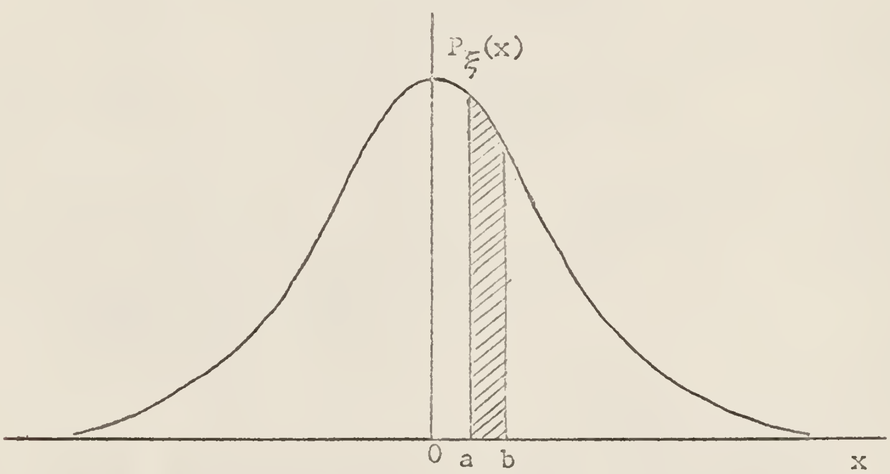


Figure 1. Probability density in normal distribution with zero mean.

3. Correlation Functions

A further refinement in describing the statistical characteristics of a random function is the autocorrelation function

$$\phi_{ff}(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(t)\, f(t+\tau)\, dt \tag{9}$$

The autocorrelation function gives some measurement of the extent to which a future value of a quantity depends on the present value. It is important to note that by setting τ equal zero, equation (9) gives the mean square value of the random quantity

$$\phi_{ff}(o) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f^2(t)\, dt = \overline{f(t)}^2 \tag{10}$$

The crosscorrelation function results when two different functions are multiplied in an expression identical in form to equation (9)

$$\phi_{fg}(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(t)\, g(t+\tau)\, d\tau \tag{11}$$

4. Power Density Spectrum

The random waveform that has been described thus far in terms of its time variable may also described in terms of its frequency spectrum. (This is strictly true only for random functions which do not change their statistical characteristics with time.) A useful function for this purpose is

the power density spectrum

$$\Phi_{ff}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_{ff}(\tau) e^{-j\omega\tau} \, d\tau \tag{12}$$

where

$$\phi_{ff}(\tau) = \int_{-\infty}^{\infty} \Phi_{ff}(\omega) e^{j\omega\tau} \, d\omega \tag{13}$$

A linear stable system of gain $H(\omega)$ with an input power density spectrum $\Phi_{ii}(\omega)$ will have an output power density

$$\Phi_{oo}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_{oo}(\tau) e^{-j\omega\tau} \, d\tau$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-j\omega\tau} \, d\tau \cdot \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} f_o(t) \, f_o(t+\tau) \, dt \tag{14}$$

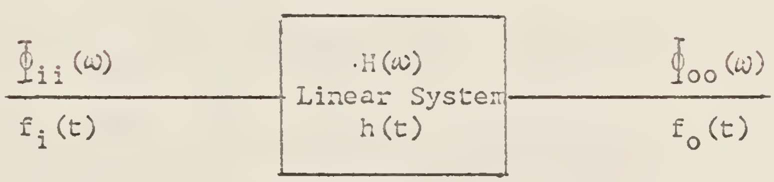


Figure 2. Linear system relationship.

Expressing the input $f_i(t)$ and $f_o(t)$ of linear system with unit impulse response $h(t)$ by a convolution integral

$$f_o(t) = \int_{-\infty}^{\infty} h(\nu)\, f_i(t - \nu)\, d\nu \tag{15}$$

Substituting equation (15) into (14), there results

$$\Phi_{oo}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-j\omega\tau}\, d\tau \cdot \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} dt \cdot \int_{-\infty}^{\infty} h(\nu)\, f_i(t - \nu)\, d\nu$$

$$\cdot \int_{-\infty}^{\infty} h(\sigma)\, f_i(t + \tau - \sigma)\, d\sigma \tag{16}$$

By inversion of the order of integration, equation (16) becomes

$$\Phi_{oo}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-j\omega\tau}\, d\tau \cdot \int_{-\infty}^{\infty} h(\nu)\, d\nu \int_{-\infty}^{\infty} h(\sigma)\, d\sigma$$

$$\cdot \lim_{T\ \infty} \frac{1}{2T}\int_{-T}^{T} f_i(t - \nu)\, f_i(t + \tau - \nu)\, dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-j\omega\tau}\, d\tau \int_{-\infty}^{\infty} h(\nu)\, d\nu \int_{-\infty}^{\infty} h(\sigma)\, d\sigma \cdot \phi_{ii}(\tau + \nu - \sigma) \tag{17}$$

With the change of variable $\mu = \tau + \nu - \sigma$ followed by separation of variables, equation (17) becomes

$$\Phi_{oo}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-j\omega(\mu+\sigma-\nu)}d\mu \int_{-\infty}^{\infty} h(\nu)\, d\nu \int_{-\infty}^{\infty} h(\sigma)d\sigma\, \phi_{ii}(\mu)$$

$$= \left[\int_{-\infty}^{\infty} h(\nu)\, e^{j\omega\nu}d\nu\right]\left[\int_{-\infty}^{\infty} h(\sigma)\, e^{-j\omega\sigma}d\sigma\right]\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} \phi_{ii}(\mu)\, e^{-j\omega\mu}d\mu\right]$$

$$= H^{*}(\omega)\cdot H(\omega)\cdot \Phi_{ii}(\omega)$$

$$= |H(\omega)|^2\, \Phi_{ii}(\omega) \tag{18}$$

By summing the power density spectrum over the entire range of frequencies, the mean square value of the random variable is obtained

$$\phi_{ff}(o) = \int_{-\infty}^{\infty} \Phi_{ff}(\omega)\, d\omega \tag{19}$$

5. Cross Power Density Spectrum

Another useful function of a random waveform that may be described in terms of frequency spectrum is the cross power density spectrum

$$\Phi_{fg}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi_{fg}(\tau)\, e^{-j\omega\tau}d\tau \tag{20}$$

where

$$\phi_{fg}(\tau) = \int_{-\infty}^{\infty} \Phi_{fg}(\omega)\, e^{j\omega\tau}d\omega \tag{21}$$

A linear stable system of gain $H(\omega)$, with input $f_i(t)$ and output $f_o(t)$, has the input-output cross-correlation function

$$\phi_{io}(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f_i(t)\, f_o(t+\tau)\, dt \tag{22}$$

To bring the system unit impulse response into this expression $f_o(t+\tau)$ can be expressed by the convolution integral

$$f_o(t+\tau) = \int_{-\infty}^{\infty} h(\nu) \cdot f_i(t+\tau-\nu)\, d\nu \tag{23}$$

Substituting equation (23) into (22), results in

$$\phi_{io}(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f_i(t)\, dt \int_{-\infty}^{\infty} h(\nu)\, f_i(t+\tau-\nu)\, d\nu$$

$$= \int_{-\infty}^{\infty} h(\nu)\, d\nu \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f_i(t)\, f_i(t+\tau-\nu)\, dt$$

$$= \int_{-\infty}^{\infty} h(\nu)\phi_{ii}(\tau-\nu)\, d\nu \tag{24}$$

Equation (24) shows that the input-output crosscorrelation of a linear system is the convolution of the unit impulse response and the input auto-correlation function.

This important relation can also be expressed in the frequency domain by a transformation

$$\Phi_{io}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi_{io}(\tau)\, e^{-j\omega\tau} d\tau$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-j\omega\tau} d\tau \int_{-\infty}^{\infty} h(\nu)\,\phi_{ii}(\tau-\nu)\, d\nu \tag{25}$$

With the change of variable, $\sigma=\tau-\nu$, equation (25) becomes

$$\Phi_{io}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-j\omega(\sigma+\nu)} d\sigma \int_{-\infty}^{\infty} h(\nu)\,\phi_{ii}(\sigma)\, d\nu$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} h(\nu)\, e^{-j\omega\nu} d\nu \int_{-\infty}^{\infty} \phi_{ii}(\sigma)\, e^{-j\omega\sigma} d\sigma$$

$$= \left[\int_{-\infty}^{\infty} h(\nu)\, e^{-j\omega\nu} d\nu\right]\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} \phi_{ii}(\sigma)\, e^{-j\omega\sigma} d\sigma\right]$$

$$= H(\omega)\cdot\Phi_{ii}(\omega) \tag{26}$$

6. Graphical Determination of Autocorrelation Function

The calculation of the approximate autocrosselation curve from experimental data is frequently desirable. To reduce equation (9) to one requiring only discrete operations, f is divided into N equally spaced small intervals as shown in Figure 3, such that each division corresponds to $\frac{2T}{N}$ seconds, if t is in seconds. The curve $f(t+\tau_m)$ is f(t) displaced m divisions to the left, corresponding to a shift of $\tau_m=m(2T/N)$ seconds. At a point n divisions n(2T/N) seconds from the origin of f(t), the value of

the curve is x_n and the value of $f(t+\tau_m)$ is x_{n+m}. The approximate expression for (9) at discrete values τ_m of τ is therefore

$$\phi_{ff}(\tau_m) = \frac{1}{N+1} \sum_{n=o}^{n-m} x_n \, x_{n+m} \tag{27}$$

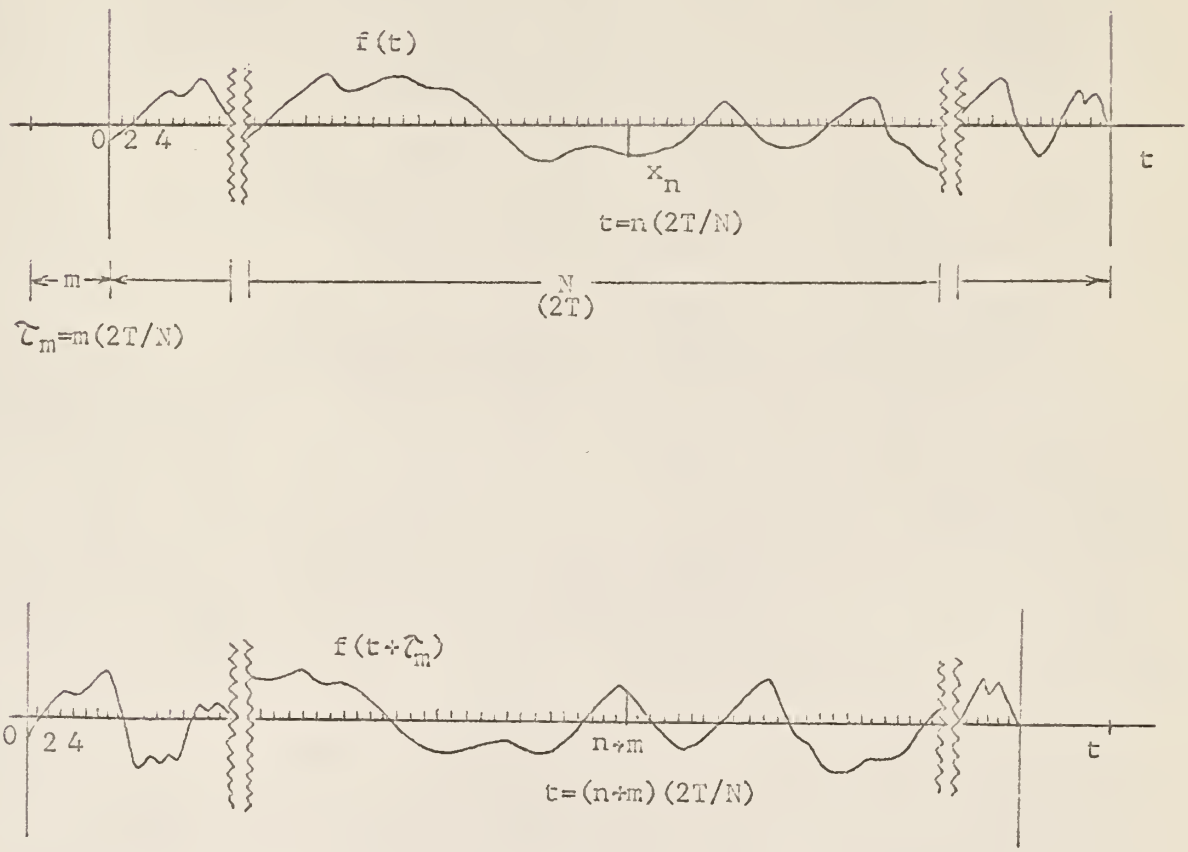


Figure 3. Graphical determination of autocorrelation curve.

NONLINEAR SYSTEM DESCRIBING FUNCTION

When a nonlinear system is subjected to random inputs with known probability density functions, some techniques of determining describing function should be investigated. This chapter states briefly how the nonlinear describing functions can be determined.

1. General Description of Booton's Method. (ref. 1)

The effect of nonlinear elements on stationary random input signals has been investigated in recent years by a number of researchers. Moreover, R. C. Booton and others have developed methods of analysis directly applicable to control system design. The Booton method characterizes the nonlinear elements by a describing function.

Using Booton's method, one finds an equivalent gain function for a nonlinear device by comparing its actual output with the output of an equivalent gain function. The chosen value for the equivalent gain is that which minimizes the rms difference between the two outputs.

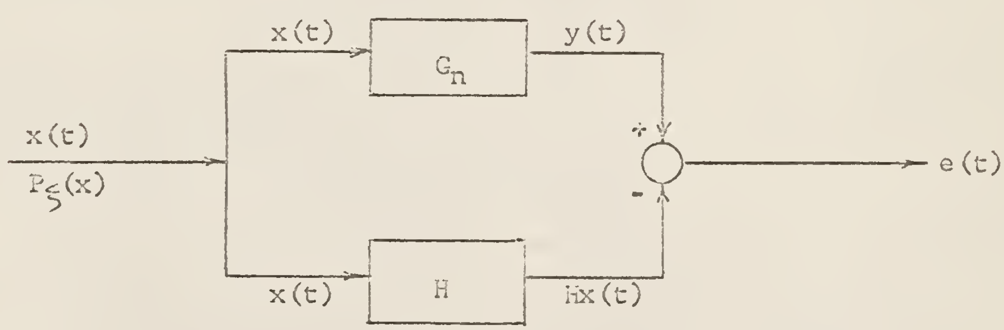


Figure 4. Booton's method.

In Figure 4, $P_{\xi}(x)$ is input probability density, G_n is a nonlinear function, H is an equivalent gain, and e is the difference between the actual gain and equivalent gain output.

From Figure 4

$$e(t) = y(t) - Hx(t) \tag{28}$$

By squaring both sides of equation (28), there is obtained

$$e^2(t) = y^2(t) - 2y(t)\,Hx(t) + (Hx(t))^2 \tag{29}$$

thus , the mean square value of e(t) is

$$\overline{e^2}(t) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} e^2(t)\,dt$$

$$= \lim_{T\to\infty} \frac{1}{2T} \left\{ \int_{-T}^{T} y^2(t)\,dt - 2H \int_{-T}^{T} y(t)\,x(t)\,dt \right.$$

$$\left. + H^2 \int_{-T}^{T} x^2(t)\,dt \right\}$$

$$= \overline{y^2} - 2H\overline{yx} + H^2\overline{x^2} \tag{30}$$

The desired value of H is the one which minimizes $\overline{e^2}(t)$, since

$$\frac{d\overline{e^2}}{dH} = -2\overline{yx} + 2H\overline{x^2} \tag{31}$$

therefore, the desire equivalent gain, H, is

$$H = \frac{\overline{yx}}{\overline{x^2}} . \tag{32}$$

Since both y(t) and x(t) are assumed to be random functions of one sort or another, the time average of equation (5) can be put into terms of probability characteristics if y(t) and x(t) are stationary processes. When the processes are ergodic, the ensemble average of the variables will be equal to time average. From equation (6), the ensemble average of x^2 is

$$\overline{x^2} = \int_{-\infty}^{\infty} x^2 P_{\xi}(x)\, dx \tag{33}$$

To find the average of yx, assume that the actual transfer characteristic of the nonlinear element is such that the output is a single valued function of the input, that is, $y(t) = f\left[x(t)\right]$, therefore

$$\overline{yx} = \int_{-\infty}^{\infty} xf(x)\; P_{\xi}(x)\, dx \tag{34}$$

Upon substituting equation (33) and (34) into equation (32), there results

$$H = \frac{\int_{-\infty}^{\infty} xf(x)\; P_{\xi}(x)\, dx}{\int_{-\infty}^{\infty} x^2 P_{\xi}(x)\, dx} \tag{35}$$

2. Describing Function For Simple Isolated Nonlinear Elements.

When the random input describing function for isolated, simple nonlinearities is a pure gain H, then, by equation (35), the describing

function can be determined when the probability density of input is known. Several nonlinearities will be discussed here before proceeding to the more general case.

a. Limiting (ref. 2)

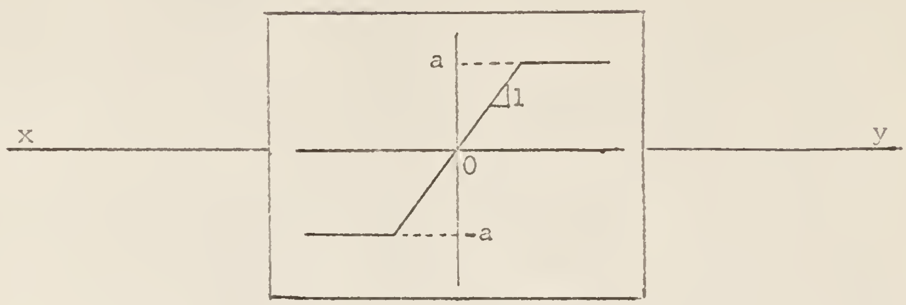


Figure 5. Input putput relation of simple nonlinearity.

In Figure 5, is a time function with a normal amplitude distribution of zero mean, so that

$$P_{\xi}(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}} \tag{36}$$

where σ is the standard deviation.

The output y(t) is

$$\begin{aligned} y(t) &= -a \;; && \text{as } x < -a \\ &= x \;; && \text{as } -a < x < a \\ &= a \;; && \text{as } x > a \end{aligned} \tag{37}$$

From equation (35), there is

$$H = \frac{\displaystyle\int_{-\infty}^{\infty} xf(x)\ P_\xi(x)\ dx}{\displaystyle\int_{-\infty}^{\infty} x^2\ P_\xi(x)\ dx} = \frac{\displaystyle\int_{-\infty}^{\infty} xf(x)\ P_\xi(x)\ dx}{2\sigma^2} \tag{38}$$

Since the integrand is an even function of x

$$\int_{-\infty}^{\infty} xf(x)\ P_\xi(x)\ dx = \int_{-\infty}^{a} -axP_\xi(x)dx + \int_{-a}^{a} x^2 P_\xi(x)dx + \int_{a}^{\infty} axP_\xi(x)dx$$

$$= 2a\int_{a}^{\infty} P_\xi(x)dx + 2\int_{0}^{a} x^2 P_\xi(x)dx$$

$$= 2a\int_{a}^{\infty} \frac{xe^{-x^2/2\sigma^2}}{\sigma\sqrt{2\pi}}\ dx + 2\int_{0}^{a} \frac{x^2 e^{-x^2/2\sigma^2}}{\sigma\sqrt{2\pi}}\ dx \tag{39}$$

Let

$$u = \frac{x^2}{2\sigma^2}\ ; \qquad du = \frac{xdx}{\sigma^2}\ ; \qquad u\Big|_{x=a} = \frac{a^2}{2\sigma^2}$$

then,

$$2a\int_{a}^{\infty} \frac{xe^{-x^2/2\ 2}}{\sigma\sqrt{2\pi}}\ dx = \frac{2a\sigma}{\sqrt{2\pi}} \int_{\frac{a^2}{2\sigma^2}}^{\infty} e^{-u}du$$

$$= \sqrt{\frac{2}{\pi}}\ a\sigma e^{-a^2/2\sigma^2}\ . \tag{40}$$

Let

$$z = \frac{x}{\sqrt{2}\sigma}\ ; \qquad dz = \frac{dx}{\sqrt{2}\sigma}\ ; \qquad z\Big|_{x=a} = \frac{a}{\sqrt{2}\sigma}$$

then,

$$2\int_0^a \frac{x^2 e^{-x^2/2\sigma^2}}{\sigma\sqrt{2\pi}}\,dx = 2\int_0^a \left(\frac{x^2}{2\sigma^2}\right)e^{-\frac{x^2}{2\sigma^2}}\frac{dx}{\sqrt{2}\sigma}\left(\frac{2\sigma^2}{\sqrt{\pi}}\right)$$

$$= \frac{4\sigma^2}{\sqrt{\pi}}\int_0^{\frac{a}{\sqrt{2}\sigma}} z^2 e^{-z^2}\,dz \tag{41}$$

Upon combining the results there is obtained

$$2a\int_a^\infty \frac{xe^{-x^2/2\sigma^2}}{\sigma\sqrt{2\pi}}\,dx + 2\int_0^a \frac{x^2 e^{-x^2/2\sigma^2}}{\sigma\sqrt{2\pi}}\,dx$$

$$= \sqrt{\frac{2}{\pi}\sigma^2}\left\{\left(\frac{a}{2\sigma}\right)e^{-a^2/2\sigma^2} + 2\int_0^{\frac{a}{2\sigma}} z^2 e^{-z^2}\,dz\right\}$$

$$= \frac{2\sigma^2}{\sqrt{\pi}}\int_0^{\frac{a}{\sqrt{2}\sigma}} e^{-z^2}\,dz \tag{42}$$

The normal input, pure gain describing function for limiting is, therefore,

$$H = \frac{\frac{2\sigma^2}{\sqrt{\pi}}}{\sigma^2}\int_0^{\frac{a}{\sqrt{2}\sigma}} e^{-z^2}\,dz = \frac{2}{\sqrt{\pi}}\int_0^{\frac{a}{\sqrt{2}\sigma}} e^{-z^2}\,dz \tag{43}$$

b. Sgn function (ref. 2)

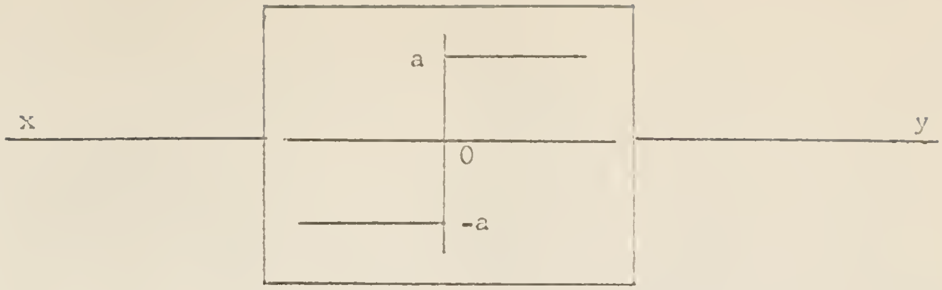


Figure 6. Input output relation of simple nonlinearity.

Let x represents a time function with a normal amplitude distribution of zero mean, so that

$$P_\xi(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-x^2/2\sigma^2} \tag{44}$$

The output y(t) is

$$\begin{aligned} y(t) &= a \; ; && \text{as } x > 0 \\ &= -a \; ; && \text{as } x > 0 \end{aligned} \tag{45}$$

From equation (35)

$$H = \frac{\int_{-\infty}^{\infty} x f(x)\, P_\xi(x)dx}{\int_{-\infty}^{\infty} x^2 P_\xi(x)dx} = \frac{-\int_{-\infty}^{0} axP_\xi(x)dx + \int_{0}^{\infty} axP_\xi(x)dx}{\sigma^2}$$

$$= \frac{2a \int_0^\infty xP_\xi(x)dx}{\sigma^2} \tag{46}$$

where

$$2a \int_{o}^{\infty} xP_{\xi}(x)dx = 2a \int_{o}^{\infty} \frac{xe^{-x^2/2\sigma^2}}{\sigma\sqrt{2\pi}} dx \tag{47}$$

Let

$$u = \frac{x^2}{2\sigma^2} ; \qquad du = \frac{xdx}{\sigma^2}$$

then,

$$2a \int_{o}^{\infty} \frac{xe^{-x^2/2\sigma^2}}{\sigma\sqrt{2\pi}} dx = \frac{2\sigma a}{\sqrt{2\pi}} \int_{o}^{\infty} e^{-u} du = \frac{2a\sigma}{\sqrt{2\pi}} \left[-e^{-u} \right]_{o}^{\infty}$$

$$= \sqrt{\frac{2}{\pi}}\, a\sigma \tag{48}$$

Upon combining these results, there results

$$H = \frac{\sqrt{\frac{2}{\pi}}\, a\sigma}{\sigma^2} \quad \frac{\sqrt{2}}{\sqrt{\pi}} \cdot \frac{a}{\sigma} \cong 0.8 \frac{a}{\sigma} \tag{49}$$

3. Threshold (ref. 2)

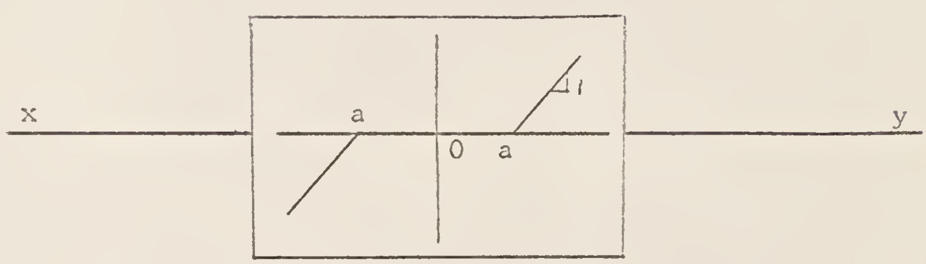


Figure 7. Input output relation of simple nonlinearity.

In Figure 7, x is a time function with a normal amplitude distribution of zero mean, so that

$$P_{\xi}(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-x^2/2\sigma^2} \tag{50}$$

The output y(t) is

$$\begin{aligned} y(t) &= (x + a) \; ; && \text{as } x < -a \\ &= 0 \; ; && \text{as } -a < x < a \\ &= (x - a) \; ; && \text{as } x > a \end{aligned} \tag{51}$$

From equation (35)

$$H = \frac{\int_{-\infty}^{\infty} x f(x) P_{\xi}(x) dx}{\int_{-\infty}^{\infty} x^2 P_{\xi}(x) dx} = \frac{\int_{-\infty}^{\infty} x f(x) P_{\xi}(x) dx}{\sigma^2}$$

$$= \frac{\int_{-\infty}^{-a} x(x + a) P_{\xi}(x) dx}{\sigma^2} + \frac{\int_{a}^{\infty} x(x - a) P_{\xi}(x) dx}{\sigma^2}$$

$$= \frac{2\int_{a}^{\infty} (x^2 - ax) P_{\xi}(x) dx}{\sigma^2} \tag{52}$$

where

$$2\int_a^\infty (x^2 - ax)P_{\xi}(x)dx = 2\int_a^\infty (x^2 - ax)\,\frac{e^{-x^2/2\sigma^2}}{\sigma\sqrt{2\pi}}\,dx$$

$$= 2\int_a^\infty \frac{x^2 e^{-x^2/2\sigma^2}}{\sigma\sqrt{2\pi}}\,dx - 2a\int_a^\infty \frac{xe^{-x^2/2\sigma^2}}{\sigma\sqrt{2\pi}}\,dx \tag{53}$$

Let

$$z = \frac{x}{\sqrt{2}\sigma}\,; \qquad dz = \frac{dx}{\sqrt{2}\sigma}\,; \qquad z\Big|_{x=a} = \frac{a}{2\sigma}$$

then,

$$2\int_a^\infty \frac{x^2 e^{-x^2/2\sigma^2}}{\sigma\; 2\pi}\,dx = 2\int_a^\infty \left(\frac{x^2}{2\sigma^2}\right)\frac{e^{-x^2/2\sigma^2}}{\sqrt{\pi}}\,dx\left(\frac{2\sigma^2}{\sqrt{2}\sigma}\right)$$

$$= \frac{4\sigma^2}{\sqrt{\pi}}\int_{\frac{a}{2\sigma}}^\infty z^2 e^{-z^2}\,dz$$

$$= \frac{2\sigma^2}{\sqrt{\pi}}\int_{\frac{a}{2\sigma}}^\infty e^{-z^2}\,dz - \frac{2\sigma^2}{\sqrt{\pi}}\,ze^{-z^2}\Big|_{\frac{a}{2\sigma}}^\infty$$

$$= \frac{2\sigma^2}{\sqrt{\pi}}\int_{\frac{a}{\sqrt{2}\sigma}}^\infty e^{-z^2}\,dz + \sqrt{\frac{2}{\pi}}\,a\sigma e^{-\left(\frac{a}{\sqrt{2}\sigma}\right)^2} \tag{54}$$

From equation (40) there is

$$2a \int_a^{\infty} \frac{xe^{-x^2/2\sigma^2}}{\sigma\sqrt{2\pi}}\, dx = \sqrt{\frac{2}{\pi}}\, a\sigma e^{-a^2/2\sigma^2} \tag{55}$$

Therefore,

$$2 \int_a^{\infty} \frac{x^2 e^{-x^2/2\sigma^2}}{\sigma\sqrt{2\pi}}\, dx - 2a \int_a^{\infty} \frac{xe^{-x^2/2\sigma^2}}{\sigma\sqrt{2\pi}}\, dx$$

$$= \frac{2\sigma^2}{\pi} \int_{\frac{a}{\sqrt{2}\sigma}}^{\infty} e^{-z^2} dz + \sqrt{\frac{2}{\pi}}\, a\sigma e^{-\frac{a^2}{2\sigma^2}} - \sqrt{\frac{2}{\pi}}\, a\sigma e^{-\frac{a^2}{2\sigma^2}}$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \int_{\frac{a}{\sqrt{2}\sigma}}^{\infty} e^{-z^2} dz = \frac{2\sigma^2}{\sqrt{\pi}} \int_0^{\infty} e^{-z^2} dz - \frac{2\sigma^2}{\sqrt{\pi}} \int_0^{\frac{a}{\sqrt{2}\sigma}} e^{-z^2} dz$$

$$= \sigma^2 \left[1 - \frac{2}{\sqrt{\pi}} \int_0^{\frac{a}{\sqrt{2}\sigma}} e^{-y^2} dy \right] \tag{56}$$

Substituting equation (56) into (52), there results

$$H = 1 - \frac{2}{\sqrt{\pi}} \int_0^{\frac{a}{\sqrt{2}\sigma}} e^{-z^2} dz \tag{57}$$

3. Generalization of Isolated Nonlinear Element.

The basic concept of the statistical describing function has been explored for simple examples. The generalization of these procedures that leads to a linear equivalent system that is not restricted to a pure gain is derived as follows:

If the nonlinear element is approximated by some linear weighting function $h(\tau)$. The error in approximation is then,

$$e(t) = y(t) - \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \qquad (58)$$

By squaring both sides of equation (58), there results

$$e^2(t) = y^2(t) - 2y(t) \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau + \left[\int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau\right] \cdot \left[\int_{-\infty}^{\infty} h(u)x(t-u)du\right] \qquad (59)$$

Taking the mean value of of equation (59), there is

$$\overline{e^2(t)} = \lim_{T\to\infty} \frac{1}{2T}\left\{ \int_{-T}^{T} y^2(t)dt - 2\int_{-T}^{T} y(t)\left[\int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau\right]dt + \int_{-T}^{T}\left[\int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau\right]\left[\int_{-\infty}^{\infty} h(u)x(t-u)du\right]dt\right\} \qquad (60)$$

Interchanging the order of integration, equation (60) becomes

$$\overline{e^2(t)} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} y^2(t)dt - 2 \int_{-\infty}^{\infty} h(\tau) \left[\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} y(t)x(t - \tau)dt \right] d\,\tau$$

$$+ \int_{-\infty}^{\infty} h(\tau)d\tau \int_{-\infty}^{\infty} h(u)du \cdot \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t - \tau)x(t - u)dt$$

(61)

Now

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} y^2(t)dt = \phi_{yy}(0)$$

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} y(t)x(t - \tau)dt = \phi_{yx}(-\tau) = \phi_{xy}(\tau) \qquad (62)$$

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t - \tau)x(t - u)dt = \phi_{xx}(\tau - u)$$

Substituting equation (62) into (61), there results

$$\overline{e^2(t)} = \phi_{yy}(0) - 2 \int_{-\infty}^{\infty} h(\tau)\phi_{xy}(\tau)d \quad + \int_{-\infty}^{\infty} d\,\tau \int_{-\infty}^{\infty} h(\tau)h(u)\phi_{xx}(\tau - u)du$$

(63)

Applying the calculus of variations, the weighting function $h(\tau)$ will be set equal to $w(\tau) + \lambda z(\tau)$, where $w(\tau)$ is the optimum weighting function, λ is the small multiplier independent of τ, and $z(\tau)$ is a differentiable function of τ. Equation (63), then, becomes

$$\overline{e^2(t)} = \phi_{yy}(0) - 2 \int_{-\infty}^{\infty} \left[w(\tau) + \lambda z(\tau)\right] \phi_{xy}(\tau) d\tau$$

$$+ \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} du \left[w(\tau) + \lambda z(\tau)\right] \left[w(u) + z(u)\right] \phi_{xx}(\tau - u)$$

(64)

Differentiating equation (64) with respect to λ, and setting the expression equal to zero, equation (64) becomes

$$\frac{d\overline{e^2(t)}}{d\lambda} = -2 \int_{-\infty}^{\infty} z(\tau)\phi_{xy}(\tau)d\tau + \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} du z(\tau) \left[w(u) + z(u)\right]$$

$$\cdot \phi_{xx}(\tau - u) + \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} du z(u) \left[w(\tau) + \lambda z(\tau)\right] \phi_{xx}(\tau - u)$$

$$= 0 \tag{65}$$

Setting $\lambda = 0$, equation (65) becomes

$$-2 \int_{-\infty}^{\infty} z(\tau)\phi_{xy}(\tau)d\tau + \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} du z(\tau)w(u)\phi_{xx}(\tau - u)$$

$$+ \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} d\ z(u)w(\tau)\phi_{xx}(u - \tau) = 0$$

or

$$-2 \int_{-\infty}^{\infty} z(\tau)\left[\phi_{xy}(\tau) - \int_{-\infty}^{\infty} \phi_{xx}(\tau - u)w(u)du\right] d\tau = 0 \qquad (66)$$

Therefore,

$$\phi_{xy}(\tau) = \int_{-\infty}^{\infty} \phi_{xx}(\tau - u)w(u)du \qquad (67)$$

Replacing $w(\tau)$ by $h(u)$, equation (67) becomes

$$\phi_{xy}(\tau) = \int_{-\infty}^{\infty} h(u)\phi_{xx}(\tau - u)du \qquad (68)$$

The result of equation (68) has shown that the input-output crosscorrelation of the best linear equivalent system is the convolutinn of the linear equivalent weighting function and the input autocorrelation.

This important relationship can be transformed into frequency domain as from equation (24) to (26), therefore,

$$\Phi_{xy}(\omega) = H(\omega)\Phi_{xx}(\omega) \qquad (69)$$

$$H(\omega) = \frac{\Phi_{xy}(\omega)}{\Phi_{xx}(\omega)} \qquad (70)$$

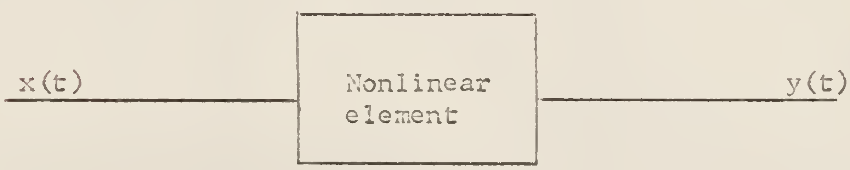


Figure 8. Isolated nonlinear element with a random input.

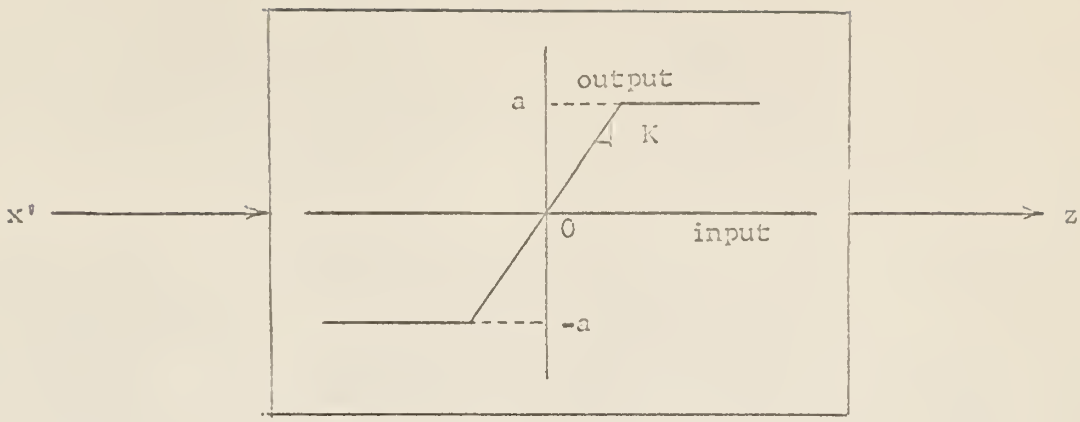
4. Another Method of Determining Random Input Describing Function

Another method of defining the random input describing function has been suggested by N. P. Pastel. (ref. 3) Passing a random signal through a nonlinear device changes its probability distribution. For example, Figure 10 shows the alteration of the probability distribution in a particular case, where the nonlinear element is a saturating amplifier. The amplifier is linear with gain K for output signal amplitudes between $\pm$a. The saturating phenomenon can be represented by a linear amplifier followed by a limiter. This is shown in Figure 9a and 9b.

Assume a normally distributed input signal, x, passes through the saturating amplifier. Since the linear amplifier does not change the form of distribution, the output of linear amplifier, x, will have the probability density as shown in Figure 10a. When this signal is passed through the limiter, the distribution remains unchanged between $\pm$ a. But the output cannot exceed $\pm$ a, and thus the probability of obtaining an output with absolute magintude greater than a is zero. On the other hand, all of the large input signals cause a saturated output, so the probability of obtaining an output of amplitude +a (or -a) is greatly increased, and must be equal to the area under the input probability curve from +a (or $-\infty$) to $+\infty$ (or -a). This output condition is represented on the probability curve as a pair of vertical lines, or impulses at equal to $\pm$ a.

The area under the output distribution curve is indicated in Figure 10b. Thus the total area under the output distribution curve is unity as required for a probability dendity function. Figure 11 shows output probability density functions for limiting, sgn, and threshald function.

a.



b.

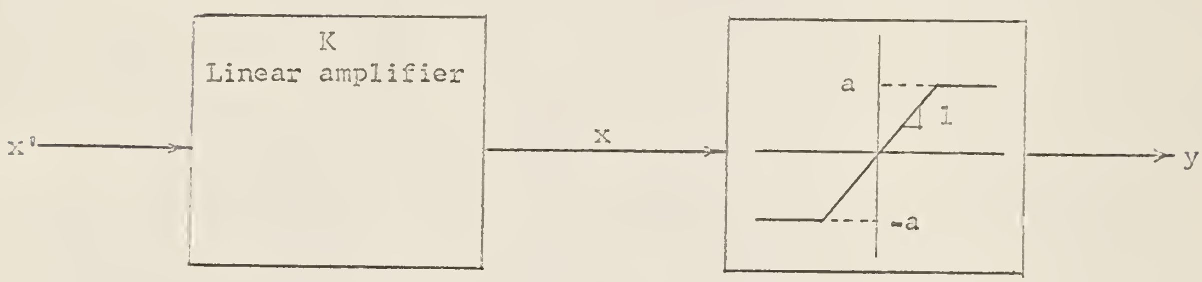
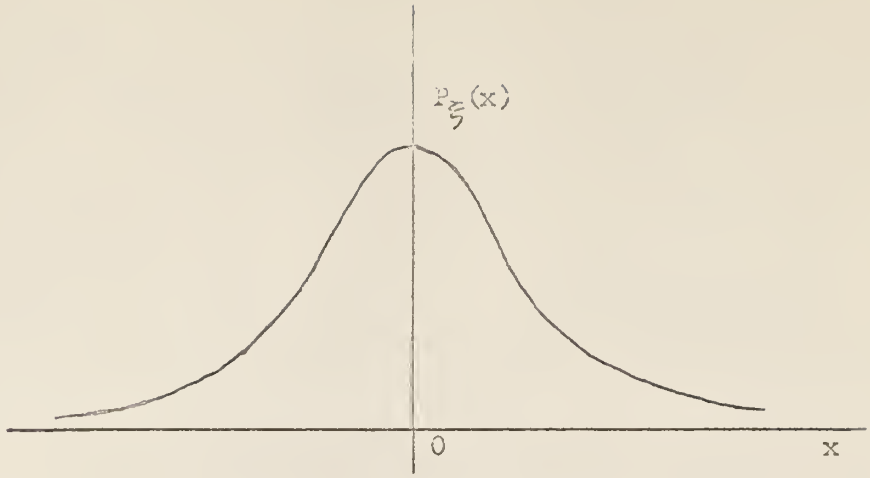


Figure 9a. Saturation amplifier response.

b. Equivalent system of saturating amplifier.

a.



b.

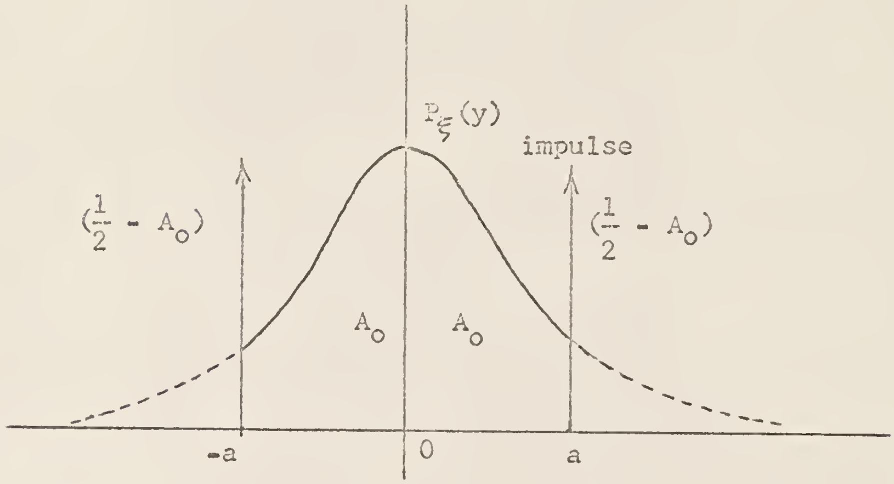


Figure 10a. Input probability density.

b. Output probability density of a limiter .

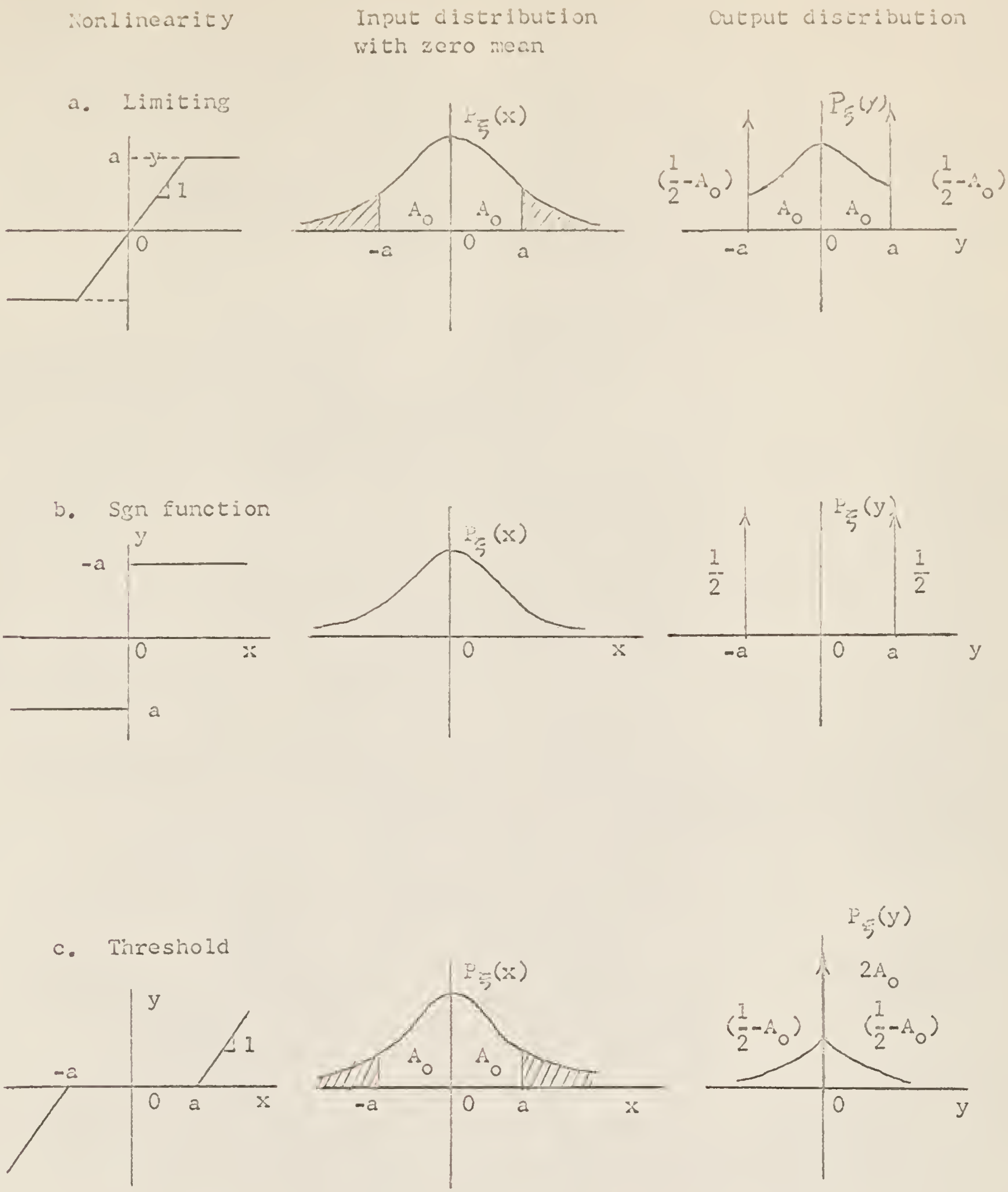


Figure 11. Distribution function for common nonlinearities. (ref. 3)

In practical applications only normally distributed inputs are usually considered, for they remain normally distributed when passed through a linear network or when added to another normally distributed random signal.

Assuming a normally distributed input with zero mean passed through some simple nonlinearities, the procedure for evaluating the statistical describing function is illustrated below.

a. Limiting

As in Fig. 5, x represents an input time function with a normal amplitude distribution of zero mean, and y is the time function of the output. Then, referring to Figure 11a, the variance or the mean square value of the output distribution is seen to be

$$\sigma_y^2 = 2 \int_0^a x^2 P_\xi(x)dx + 2a^2\left[\frac{1}{2} - \int_0^a P_\xi(x)dx\right] \tag{71}$$

where

$$P_\xi(x) = \frac{e^{-x^2/2\sigma^2}}{\sigma\sqrt{2\pi}} \tag{72}$$

Substituting equation (72) into (71), yields

$$\sigma_y^2 = \frac{2}{\sigma\sqrt{2\pi}} \int_0^a x^2 e^{-x^2/2\sigma^2} dx + 2a^2\left[\frac{1}{2} - \frac{1}{\sigma\sqrt{2\pi}} \int_0^a e^{-x^2/2\sigma^2} dx\right] \tag{73}$$

Using integration by parts, equation (73) becomes

$$\sigma_y^2 = \frac{1}{\sigma\sqrt{2\pi}} \left\{\left[-2\sigma^2 x e^{-\frac{x^2}{2\sigma^2}}\right]_0^a + 2\sigma^2 \int_0^a e^{-\frac{x^2}{2\sigma^2}} dx\right\}$$

$$+ 2a^2 \left[\frac{1}{2} - \frac{1}{\sigma\sqrt{2\pi}} \int_0^a e^{-x^2/2\sigma^2} dx\right] \tag{74}$$

Let

$$A_o = \frac{1}{\sigma\sqrt{2\pi}} \int_o^a e^{-x^2/2\sigma^2}\, dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_o^a \left(1 - \frac{x^2}{2\sigma^2} + \frac{x^4}{8\sigma^4} - \frac{x^6}{48\sigma^6} + \frac{x^8}{384\sigma^8} + \cdots\right) dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\left(\frac{a}{\sigma}\right) - \frac{1}{6}\left(\frac{a}{\sigma}\right)^3 + \frac{1}{40}\left(\frac{a}{\sigma}\right)^5 - \frac{1}{336}\left(\frac{a}{\sigma}\right)^7 + \cdots \right] \qquad (75)$$

then, equation (74) becomes

$$\sigma_y^2 = \frac{-2\sigma}{\sqrt{2\pi}}\, a e^{-\frac{1}{2}\left(\frac{a}{\sigma}\right)^2} + \sigma^2 A_o + 2a^2\left(\frac{1}{2} - A_o\right)$$

or

$$\frac{\sigma_y^2}{\sigma^2} = \frac{-2}{2\pi}\left(\frac{a}{\sigma}\right) e^{-\frac{1}{2}\left(\frac{a}{\sigma}\right)^2} + A_o + 2\left(\frac{a}{\sigma}\right)^2\left(\frac{1}{2} - A_o\right) \qquad (77)$$

Inserting numerical values into equation (75) and (76) and taking the square root of equation (76), the statistical describing function is

$$H_\sigma = \frac{\sigma_y}{\sigma} \qquad (77)$$

Equation (77) is plotted in graphical form as shown in Figure 12. In order to compare the two methods mentioned so far, a curve of H as given by equation (43) is also shown in Figure 12.

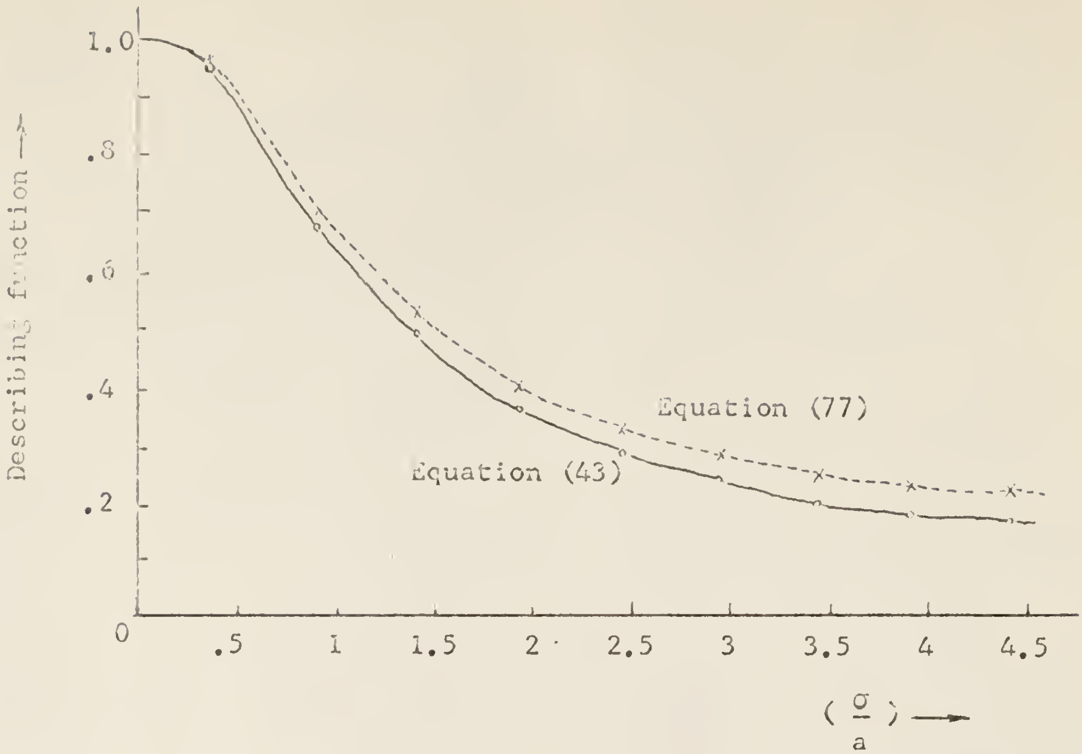


Figure 12. Value of describing function for simple limiter with normal distribution input.

b. Sgn function

As in Figure 6, x is an input time function with a normal amplitude distribution of zero mean, and y is the output time function. Referring to Figure 11b, the variance of the output distribution is seen to be

$$\sigma_y^{\,2} = 2a^2 \cdot \frac{1}{2} = a^2 \tag{78}$$

Therefore,

$$\frac{\sigma_y^{\,2}}{\sigma^2} = \left(\frac{a}{\sigma} \right)^2 \tag{79}$$

or

$$H_\sigma = \frac{a}{\sigma} \tag{80}$$

Equation (80) as well as equation (49), has been plotted on Figure 13.

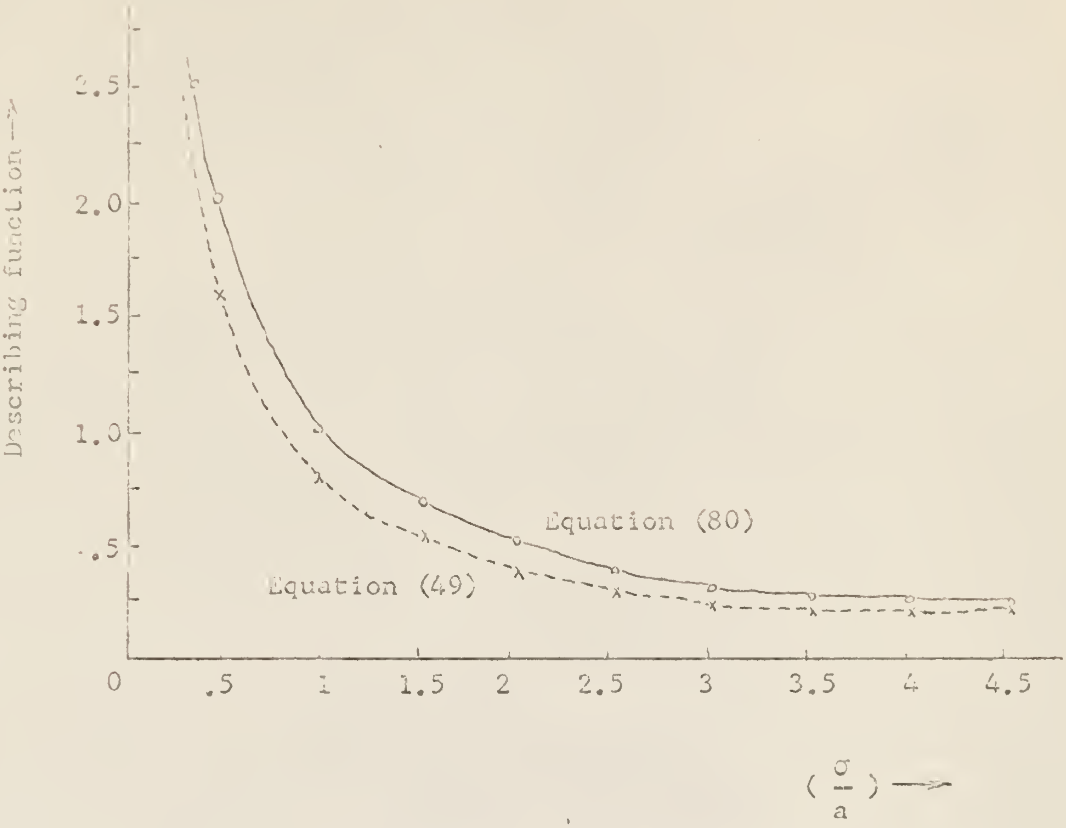


Figure 13. Value of describing function for sgn function with normal distribution input.

c. Threshold

In Figure 7, x represents an input time function, having normal distribution of zero mean, and y is the output time function. From Figure 11c, the variance of the output distribution is found to be

$$\sigma_y^2 = 2\int_a^{\infty}(x - a)^2 P_{\xi}(x)dx$$

$$= 2\int_a^{\infty}(x^2 - 2ax + a^2)P_{\xi}(x)dx$$

$$= \frac{2}{\sigma\sqrt{2\pi}} \int_a^\infty x^2 e^{-x^2/2\sigma^2}\, dx - \frac{4a}{\sigma\sqrt{2\pi}} \int_a^\infty x e^{-x^2/2\sigma^2}\, dx$$

$$+ \int_a^\infty e^{-x^2/2\sigma^2}\, dx$$

$$= \frac{2}{\sigma\sqrt{2\pi}} \left\{ \left[-\sigma^2 x e^{-\frac{x^2}{2\sigma^2}} \right]_a^\infty + \sigma^2 \int_a^\infty e^{-\frac{x^2}{2\sigma^2}}\, dx \right\}$$

$$+ \frac{4a\sigma^2}{\sigma\sqrt{2\pi}} \left[e^{-x^2/2\sigma^2} \right]_a^\infty + \frac{2a^2}{\sigma\sqrt{2\pi}} \int_a^\infty e^{-x^2/2\sigma^2}\, dx$$

$$= \frac{2\sigma^2}{\sqrt{2\pi}} \left(\frac{a}{\sigma} \right) e^{-\frac{1}{2}\left(\frac{a}{\sigma}\right)^2} + 2\sigma^2 \left(\frac{1}{2} - A_o \right) - \frac{4\sigma^2}{\sqrt{2\pi}} \left(\frac{a}{\sigma} \right) e^{-\frac{1}{2}\left(\frac{a}{\sigma}\right)^2}$$

$$+ 2a^2 \left(\frac{1}{2} - A_o \right)$$

$$= - \frac{2\sigma^2}{\sqrt{2\pi}} \left(\frac{a}{\sigma} \right) e^{-\frac{1}{2}\left(\frac{a}{\sigma}\right)^2} + 2\left[\sigma^2 + a^2 \right] \left(\frac{1}{2} - A_o \right) \tag{81}$$

$$\frac{\sigma_y^2}{\sigma^2} = - \frac{2}{\sqrt{2\pi}} \left(\frac{a}{\sigma} \right) e^{-\frac{1}{2}\left(\frac{a}{\sigma}\right)^2} + 2\left[1 + \left(\frac{a}{\sigma} \right)^2 \right] \left(\frac{1}{2} - A_o \right) \tag{82}$$

where A_o is the same as equation (75).

Inserting numerical values and taking the square root of equation (82), the

statistical describing function is

$$H_\sigma = \frac{\sigma_y}{\sigma} \qquad (83)$$

The graphical form of equation (83) and (57) are plotted on Figure 14.

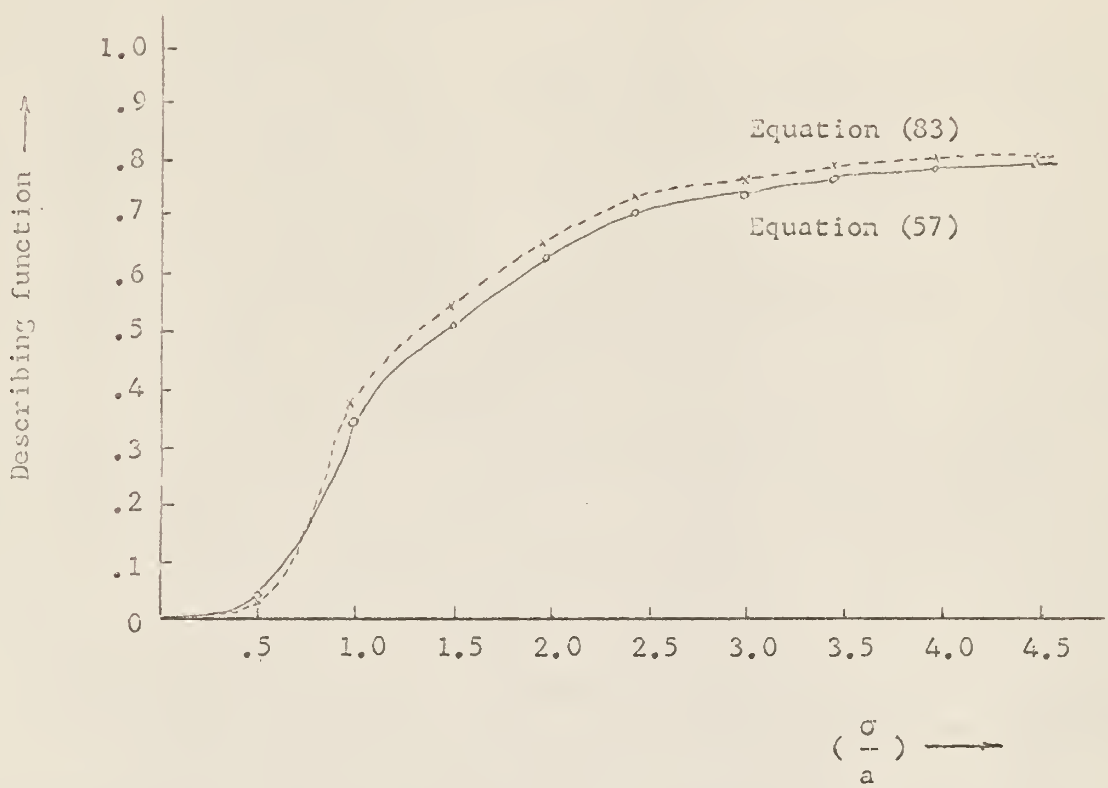


Figure 14. Value of describing function for a threhold non-linearity with normal distribution input.

THE ANALYSIS OF CLOSED LOOP SYSTEM

Because the presence of a nonlinear element in the system destroys the normally distributed nature of the signals, the input signal to the nonlinear device cannot be a normal distribution in a feedback system. However, it has been shown (ref. 4) that a relatively low system bandwidth tends to redistribute the signals into a normal form. Thus it is a practical engineering assumption to consider the return signal to the input of the nonlinear element as normally distributed. This assumption permits analysis of feedback systems in the same manner as the open loop examples given above. This chapter deals with two different methods of analyzing closed loop nonlinear systems.

1. Correlation Function Method

For the case of the isolated element shown in Figure 8, the random input discribing function according to equation (70) is

$$H(\omega) = \frac{\Phi_{xy}(\omega)}{\Phi_{xx}(\omega)} \tag{84}$$

For the prototype closed loop control system of Figure 15, the describing function, H, represents the nonlinear element, while $G(\omega)$ describes the performence of the linear constant-parameter elements of the system.

From Figure 15, the signals within the system expressed in the frequency domain are

$$E(\omega) = I(\omega) - C(\omega) = I(\omega) - G(\omega).M(\omega) \tag{85}$$

$$M(\omega) = H(\omega) \cdot E(\omega) \tag{86}$$

These equations may be rearranged to yield

$$E(\omega) = \frac{I(\omega)}{1 + G(\omega)H(\omega)} \tag{87}$$

$$M(\omega) = \frac{H(\omega)I(\omega)}{1 + G(\omega)H(\omega)} \tag{88}$$

$$C(\omega) = \frac{G(\omega)H(\omega)I(\omega)}{1 + B(\omega)H(\omega)} \tag{89}$$

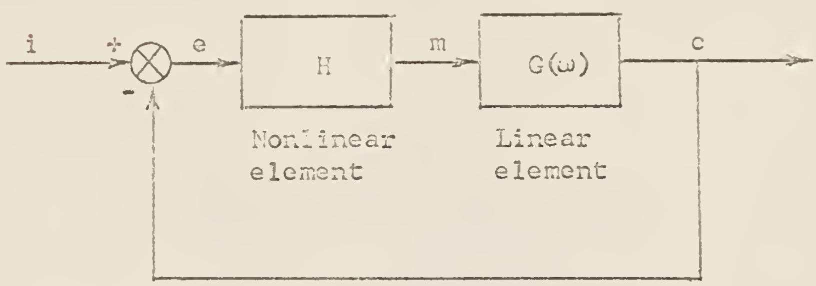


Figure 15. Prototype closed loop control system with random input.

Now according to equation (26)

$$\frac{E(\omega)}{I(\omega)} = \frac{\Phi_{ie}}{\Phi_{ii}} \tag{90}$$

$$\frac{M(\omega)}{I(\omega)} = \frac{\Phi_{im}}{\Phi_{ii}} \tag{91}$$

$$\frac{C(\omega)}{I(\omega)} = \frac{\Phi_{ic}}{\Phi_{ii}} \tag{92}$$

therefore,

$$\frac{\Phi_{ie}}{\Phi_{ii}} = \frac{1}{1 + G(\omega)H(\omega)} \tag{93}$$

$$\frac{\Phi_{im}}{\Phi_{ii}} = \frac{H(\omega)}{1 + G(\omega)H(\omega)} \tag{94}$$

$$\frac{\Phi_{ic}}{\Phi_{ii}} = \frac{G(\omega)H(\omega)}{1 + G(\omega)H(\omega)} \tag{95}$$

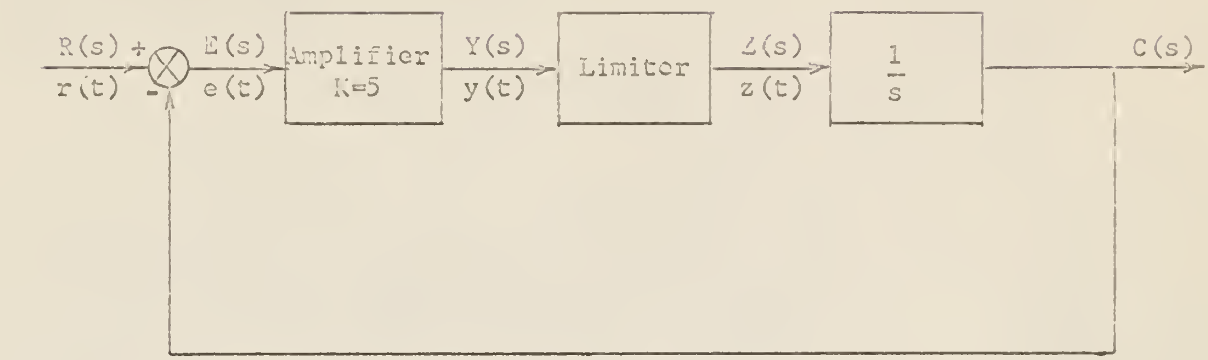
Once the power density spectrum and crosspower density spectrum are determined, the random input describing function can also be determined from equations (93) to equation (95).

2. Graphical Method

Another method of obtaining the random input describing function for a nonlinear element is described by M. P. Pastel. (ref. 3) When a nonlinear device is a component in a feed-back control system, its input is a function not only of the system command signal but also of the signals feedback around it. It would then seem that the describing function itself must be known to the input signal. Fortunately, two simultaneous equations which contain the describing function and its input as the unknown can be developed. The first is written from conventional feedback system equations using the given power density spectrum as the system input and stating the describing function as an unknown parameter. The second is generated graphically, such as in Figure 12 (for the case of simple limiter), from the output distribution form of describing function. A plot of the first equation on the graph of the second will then give the solution for the value of the

describing function at the intersection of the two curves. This procedure is illustrated by a simple example below.

a.



b.

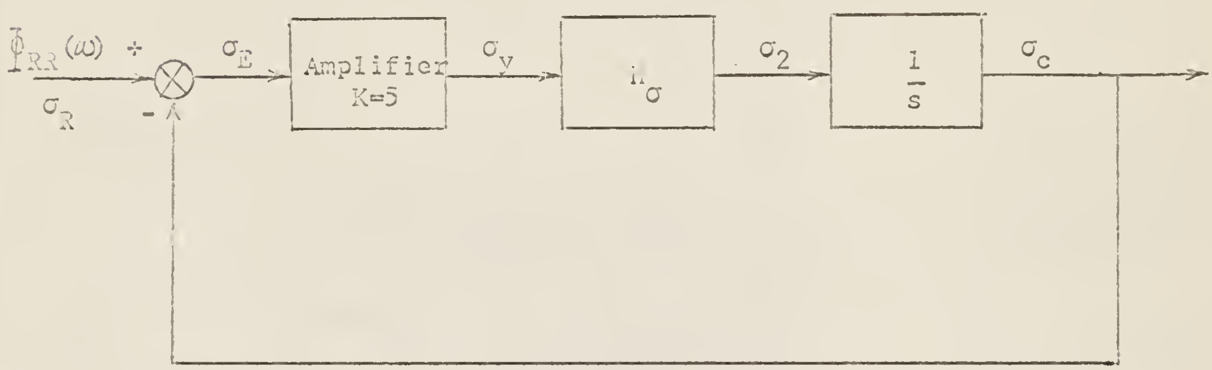


Figure 16a. Nonlinear servo system.

b. System in a form for statistical analysis.

The position servo in Figure 16a contains a saturating element, and is subjected to a random input with normal distribution and power density spectrum $\Phi_{RR}(\omega)$. The first step in the analysis is shown in Figure 16b, in which the limiter has been replaced by its describing function H_σ. From Figure 16a there is obtained

$$\frac{C(s)}{E(s)} = \frac{5H_\sigma}{s}$$

$$\frac{C(s)}{R(s)} = \frac{5H_\sigma}{s + 5H_\sigma}$$

$$\frac{E(s)}{R(s)} = \frac{\dfrac{C(s)}{R(s)}}{\dfrac{C(s)}{E(s)}} = \frac{s}{s + 5H_\sigma}$$

where the limiter is replaced by its describing function H_σ.

Therefore, the input to limiter Y(s) as shown in Figure 16a is

$$Y(s) = 5E(s) = \frac{5s}{s + 5H_\sigma} R(s) \tag{96}$$

If the input signal has a power density spectrum of

$$\Phi_{RR} = \frac{4}{1 + \omega^2} \tag{97}$$

then the power density spectrum at the input to the limiter is

$$\Phi_{yy}(\omega) = \Phi_{RR}(\omega) \cdot \left| \frac{Y(\omega)}{R(\omega)} \right|^2$$

$$= \frac{4}{1 + \omega^2} \left| \frac{j5\omega}{j\omega + 5H_\sigma} \right|^2$$

$$= \frac{4}{(1 + j\omega)(1 - j\omega)} \cdot \frac{(j5\omega)(-j5\omega)}{(5H_\sigma + j\omega)(5H_\sigma - j\omega)}$$

$$= \frac{100\omega^2}{(\omega - j1)(\omega + j1)(\omega - j5H_\sigma)(\omega + j5H_\sigma)} \qquad (98)$$

The first relationship between the rms value of the input to the limiter and the describing function is then

$$\sigma_y^2 = \int_{-\infty}^{\infty} \tilde{\Phi}_{yy}(\omega)d\omega$$

$$= \int_{-\infty}^{\infty} \frac{100\omega^2}{(\omega - j1)(\omega + j1)(\omega - j5H_\sigma)(\omega + j5H_\sigma)} d$$

$$= 2\pi j \Big[\text{Residue at the poles of } \frac{100\omega^2}{(\omega - j1)(\omega + j1)(\omega - j5H_\sigma)(\omega + j5H_\sigma)} \text{ lie in upper half plane} \Big]$$

$$= 2\pi j \left\{ \left[\frac{100\omega^2}{(\omega - j1)(\omega + j1)(\omega + j5H_\sigma)} \right]_{\omega = j5H_\sigma} + \left[\frac{100\omega^2}{(\omega - j1)(\omega + j5H_\sigma)(\omega - j5H_\sigma)} \right]_{\omega = j1} \right\}$$

$$= 2\pi j \left\{ \frac{250H_\sigma}{j(25H_\sigma^2 - 1)} + \frac{-50}{j(25H_\sigma^2 - 1)} \right\}$$

$$= \frac{100\pi}{1 + 5H_\sigma} \qquad (99)$$

Upon taking the square root of equation (99), there results

$$\sigma_y = \frac{17.7}{\sqrt{1 + 5H_\sigma}} \tag{100}$$

The second relationship between σ_y and H is given by the curve in Figure 12, with the saturation value of the limiter defined as "a" equals 2.5. Figure 17 is the same as Figure 12 with curve added for the function of equation (100) divided by 2.5. The intersection of this two curves provides the simultaneous solution of the equations. Hence for this case, the value of σ_y = 1.825 and the value of H_σ is 0.88. The value of the rms error signal σ_E is then 0.365.

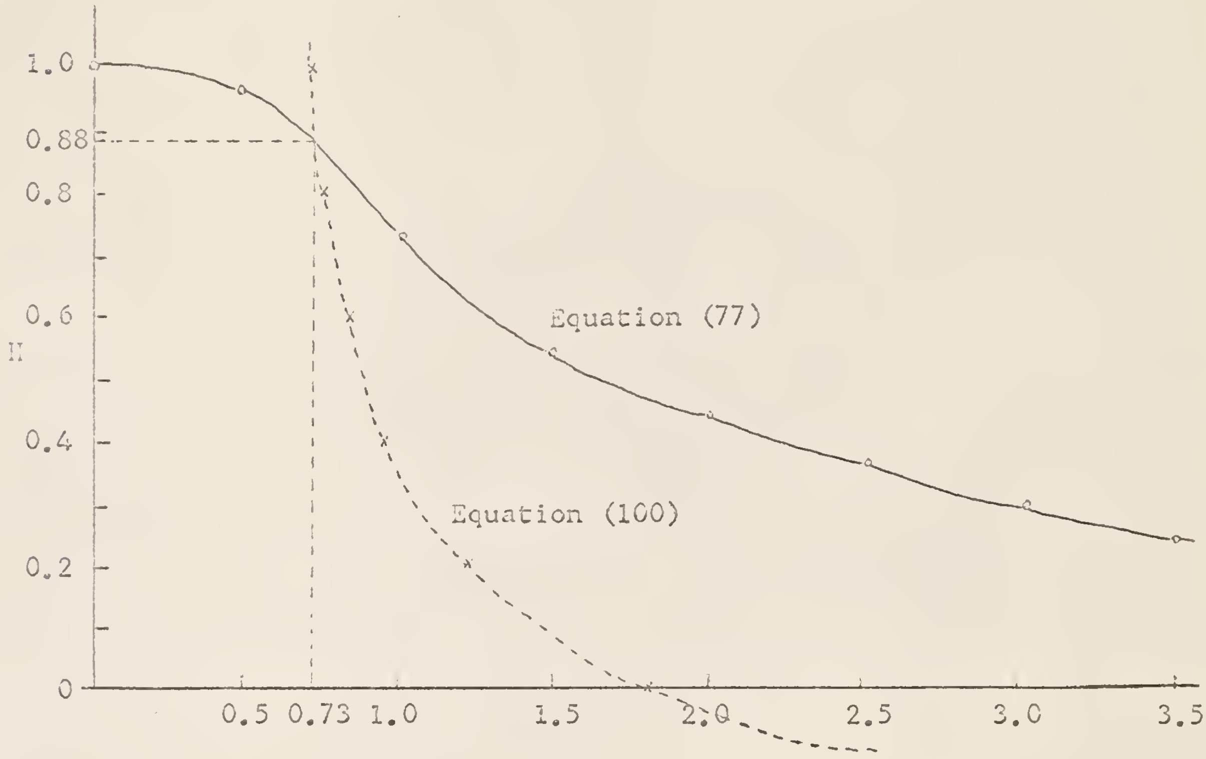


Figure 17. Graphical solution for describing function and rms value of the input to limiter in Figure 16.

THE ANALYSIS OF JUMP PHENOMENON

A rather interesting example of using the normal random input describing function is the jump resonance phenomenon. (ref. 2) In Figure 18 is shown a servomechanism with a saturating amplifier.

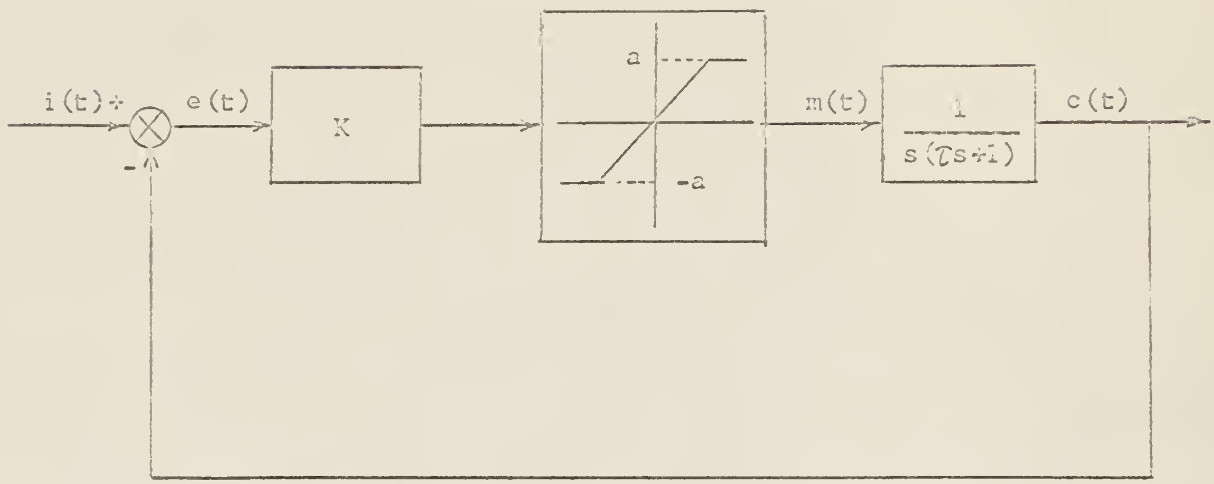


Figure 18. Block diagram of a servomechanism with a saturating amplifier.

If one supposes that the input possesses a normal amplitude probability distribution and a power spectral density which is the equivalent of the one obtained when white noise is passed through a first order lag-filter, then, according to equation (18)

$$\Phi_{ii}(\omega) = |H(\omega)|^2 \Phi_{kk}(\omega) \tag{101}$$

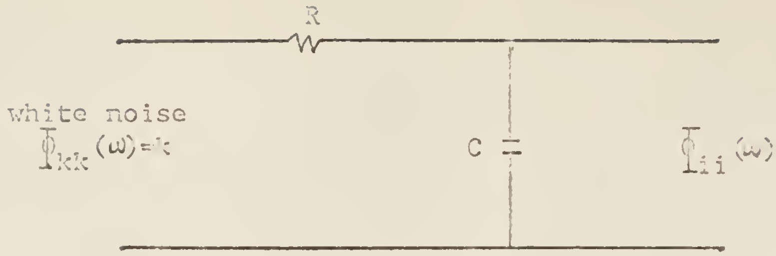


Figure 19. Input output relation of first order lag-filter.

From Figure 19, the transfer function $H(\omega)$ is

$$H(\omega) = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} = \frac{1}{jRC\omega + 1} = \frac{\frac{1}{RC}}{j\omega + \frac{1}{RC}} = \frac{\omega_i}{j\omega + \omega_i} \tag{102}$$

thus

$$|H(\omega)|^2 = \frac{\omega_i^{\,2}}{\omega^2 + \omega_i^{\,2}} \tag{103}$$

where $\omega_i = \frac{1}{RC}$ which is the break frequency of the first order lag-filter. Upon substituting equation (103) into (101), there is obtained

$$\Phi_{ij}(\omega) = k\,\frac{\omega_i^{\,2}}{\omega^2 + \omega_i^{\,2}} \tag{104}$$

Under the assumption that the input to the limiter is a stationary normal input, the limiter characteristic is replaced by its first approximation equivalent. According to equation (43) it holds that $K = \frac{2}{2\pi}\int_0^{\frac{a}{\sigma\sqrt{2}}} e^{-z^2}\,dz$, in which K is the variance sensitive pure gain. The closed loop error input transfer function is then given by

$$\frac{E(s)}{I(s)} = \frac{1}{1 + \dfrac{HK}{s(\tau s + 1)}} = \frac{s(s + \frac{1}{\tau})}{s^2 + \dfrac{s}{\tau} + \dfrac{HK}{\tau}} \tag{105}$$

The power spectral density of the error, e(t), is

$$\begin{aligned}
\Phi_{ee}(\omega) &= \left|\frac{E(\omega)}{I(\omega)}\right|^2 \Phi_{ii}(\omega) \\
&= \left|\frac{(j\omega)(j\omega + \frac{1}{\tau})}{(j\omega)^2 + \dfrac{j\omega}{\tau} + \dfrac{HK}{\tau}}\right|^2 \frac{K\omega^2}{\omega^2 + \omega_i^2} \\
&= K\left|\frac{(j\omega)(j\omega + \frac{1}{\tau})}{(j\omega)^2 + \dfrac{j\omega}{\tau} + \dfrac{HK}{\tau}}\right|^2 \left|\frac{\omega_i}{j\omega + \omega_i}\right|^2 \\
&= K\omega_i^2 \left\{ \frac{(j\omega)^4 - \frac{1}{\tau^2}(j\omega)^2}{\left|(j\omega)^3 + (\frac{1}{\tau} + \omega_i)(j\omega)^2 + \frac{1}{\tau}(HK + \omega_i)j\omega + \dfrac{hK\omega_i}{\tau}\right|^2} \right\}
\end{aligned} \tag{106}$$

The mean square value of the error, according to equation (10) and (13), is

$$\begin{aligned}
\overline{e^2(t)} &= \Phi_{ff}(0) = \int_0^\infty \Phi_{ee}(\omega)d\omega = \frac{1}{2}\int_{-\infty}^{\infty} \Phi_{ee}(\omega)d\omega \\
&= \frac{K\omega_i^2}{2}\int_{-\infty}^{\infty} \frac{(j\omega)^4 - \frac{1}{\tau^2}(j\omega)^2}{\left|(j\omega)^3 + (\frac{1}{\tau} + \omega_i)(j\omega)^2 + \frac{1}{2}(HK + \omega_i)j\omega + \dfrac{HK}{\tau}\omega_i\right|^2}
\end{aligned} \tag{107}$$

Equation (107) may be identified with integrals of the general form of

$$I_n = \frac{1}{2\pi j} \int_{-\infty}^{\infty} \frac{g_n(x)}{h_n(x)\ h_n(-x)}\, dx \tag{108}$$

where

$$h_n(x) = a_o x^n + a_1 x^{n-1} + \cdots + a_n$$

$$g_n(x) = b_o x^{2n-2} + b_1 x^{2n-4} + \cdots + b_{n-1}$$

An abbreviated table of these integral is presented in the appendix. In present case

$$\overline{e^2(t)} = (k\omega_i{}^2\pi) \cdot \frac{1}{2\pi j} \int_{-\infty}^{\infty} \frac{b_o x^4 + b_1 x^2 + b_2}{\left| a_o x^3 + a_1 x^2 + a_2 x + a_3 \right|^2}$$

$$= (k\omega_i{}^2\pi) \cdot I_3$$

where

$$a_o = 1 \qquad b_o = 1$$

$$a_1 = \left(\frac{1}{\tau} + \omega_i \right) \qquad b_1 = -\frac{1}{\tau^2}$$

$$a_2 = \frac{1}{\tau}(HK + \omega_i) \qquad b_2 = 0$$

$$a_3 = \frac{1}{\tau} HK\omega_i \qquad x = j\omega$$

From table in the appendix

$$I_3 = \frac{-a_2 b_o + a_o b_1 - \dfrac{a_o a_1 b_2}{a_3}}{2a_o(a_o a_3 - a_1 a_2)} \tag{110}$$

Substituting the appropriate values for the constants, there results

$$I_3 = \frac{-\frac{1}{\tau}(HK + \omega_i) - \frac{1}{\tau^2}}{2\left[\frac{HK\omega_i}{\tau} - \frac{1}{\tau}(\frac{1}{\tau} + \omega_i)(HK + \omega_i)\right]}$$

$$= \frac{HK + \omega_i + \frac{1}{\tau}}{2\omega_i\left(\frac{HK}{\tau\omega_i} + \omega_i + \frac{1}{\tau}\right)} \qquad (111)$$

The mean square value of the error is, then,

$$\overline{e^2(t)} = (k\omega_i^{\,2}\pi) \cdot I_3$$

$$= \frac{\pi k\omega_i}{2} \cdot \frac{\left(HK + \omega_i + \frac{1}{\tau}\right)}{\left(\frac{HK}{\tau\omega_i} + \omega_i + \frac{1}{\tau}\right)} \qquad (112)$$

Now the mean square value of the input is

$$\overline{i^2(t)} = \frac{1}{2}\int_{-\infty}^{\infty} \Phi_{ii}(\omega)d\omega$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} k\,\frac{\omega_i^{\,2}}{\omega^2 + \omega_i^{\,2}}\,d\omega$$

$$= \frac{k\omega_i^{\,2}}{2} \cdot \frac{2}{\omega_i}\left[\tan^{-1}\frac{\omega}{\omega_i}\right]_0^{\infty}$$

$$= \frac{k\omega_i^{\,2}}{\omega_i} \cdot \frac{\pi}{2}$$

$$= \frac{k\pi\omega_i}{2} \tag{113}$$

therefore,

$$\overline{e^2(t)} = \overline{i^2(t)} \; \frac{HK + \omega_i + \frac{1}{\tau}}{\frac{HK}{\tau\omega_i} + \omega_i + \frac{1}{\tau}} \tag{114}$$

The mean square value (here it is equal to variance) of the input to the limiter is related to the mean square value of the system error signal by the expression

$$\sigma^2 = K^2 \cdot \overline{e^2}$$

$$= K^2 \cdot i^2 \; \frac{HK + \omega_i + \frac{1}{\tau}}{\frac{HK}{\tau\omega_i} + \omega_i + \frac{1}{\tau}} \tag{115}$$

The characteristic equation of linear system (H=1) is

$$s^2 + \frac{1}{\tau} s + \frac{K}{\tau} = s^2 + 2\xi\omega_n s + \omega_n^{\;2} = 0 \tag{116}$$

where

$$\omega_n^{\;2} = \frac{K}{\tau}$$

$$2\xi\omega_n = \frac{1}{\tau}$$

so that

$$\xi = \frac{1}{2\tau\omega_n} = \frac{1}{2\sqrt{\tau K}}$$

$$\tau K = \frac{1}{4\xi^2} \tag{117}$$

Upon substituting equation (117) into (115), there results

$$\sigma^2 = K^2 \cdot \overline{i^2} \left[\frac{\dfrac{H}{4\xi^2} + \tau\omega_i + 1}{\dfrac{H}{\tau\omega_i \cdot 4\xi^2} + \tau\omega_i + 1} \right]$$

$$= K^2 \omega_i \tau \overline{i^2} \left[\frac{H + 4\xi^2(1 + \tau\omega_i)}{H + 4\xi^2 \tau\omega_i (1 + \tau\omega_i)} \right] \tag{118}$$

or

$$K^2 \omega_i \tau \overline{i^2} = \left[\frac{H + 4\xi^2 \tau\omega_i (1 + \tau\omega_i)}{H + 4\xi^2 (1 + \tau\omega_i)} \right] \sigma^2 \tag{119}$$

Let

$$J = K^2 \tau\omega_i \overline{i^2}$$

$$\alpha = 4\xi^2 (1 + \tau\omega_i)$$

then,

$$J = \frac{H + \tau\omega_i \alpha}{H + \alpha} \sigma^2 \tag{120}$$

Recalling that H is a function of $\frac{a}{\sigma}$, it is to be expected that there will be more than one value of e^2 for any given value of i^2 if the condition

$$\frac{dJ}{d(\frac{a}{\sigma})} = \frac{d}{d(\frac{a}{\sigma})} \left\{ \left[\frac{H + \tau\omega_i \alpha}{H + \alpha} \right] \sigma^2 \right\} = 0$$

is met. The possible multivalued relationship between the input signal and

the error signal is the source of jump phenomenon. The establishment of the necessary conditions for the jump phenomenon directly from considerations of the circumstances under which the expression for the derivative may be zero proceeds as follows,

$$\frac{1}{a^2}\cdot\frac{dJ}{d(\frac{a}{\sigma})} = \frac{d}{d(\frac{a}{\sigma})}\left\{\left[\frac{H+\tau\omega_i\alpha}{H+\alpha}\right]\left(\frac{\sigma}{a}\right)^2\right\}$$

$$= \frac{\left(\frac{a}{\sigma}\right)^2 \frac{(H+\alpha)H' - (H+\tau\omega_i\alpha)H'}{(H+\alpha)^2} - \frac{2(H+\tau\omega_i\alpha)}{H+\alpha}\left(\frac{a}{\sigma}\right)}{\left(\frac{a}{\sigma}\right)^4}$$

$$= -\frac{1}{\left(\frac{a}{\sigma}\right)^2}\cdot\frac{(H+\tau\omega_i\alpha)H'}{(H+\alpha)^2} + \frac{1}{\left(\frac{a}{\sigma}\right)^2}\cdot\frac{H'}{H+\alpha} - \frac{2}{\left(\frac{a}{\sigma}\right)^3}\cdot\frac{H+\tau\omega_i\alpha}{H+\alpha}$$

$$= 0$$

where

$$H' = \frac{dH}{d(\frac{a}{\sigma})} = \frac{2}{\sqrt{\pi}}\cdot\frac{d}{d(\frac{a}{\sigma})}\left[\int_0^{\frac{a}{\sqrt{2}\sigma}} e^{-a^2/2\sigma^2}\, d\left(\frac{a}{\sqrt{2}\sigma}\right)\right]$$

$$= \sqrt{\frac{2}{\pi}}\, e^{-a^2/2\sigma^2} \tag{122}$$

Equation (121) may be reexpressed as

$$\frac{1}{\left(\frac{a}{\sigma}\right)^2}\left[-\frac{H+\tau\omega_i\alpha}{(H+\alpha)^2}\sqrt{\frac{2}{\pi}}\, e^{-a^2/2\sigma^2} - \frac{1}{H+\alpha}\sqrt{\frac{2}{\pi}}\, e^{-a^2/2\sigma^2} + \frac{2}{\left(\frac{a}{\sigma}\right)}\cdot\frac{H+\tau\omega_i\alpha}{H+\alpha}\right]$$

$$= 0 \tag{123}$$

Because $\frac{a}{\sigma} \neq 0$, therefore,

$$\left(\frac{a}{\sigma}\right) = \frac{2\,\dfrac{H + \tau\omega_i\alpha}{H + \alpha}}{\dfrac{(1 - \tau\omega_i)}{(H + \alpha)^2}\sqrt{\dfrac{2}{\pi}}\, e^{-a^2/2\sigma^2}} = \sqrt{2\pi}\,\frac{(H + \tau\omega_i\alpha)(H + \alpha)}{\alpha(1 - \tau\omega_i)}\, e^{a^2/2\sigma^2}$$

$$= \sqrt{2\pi}\cdot\frac{H + 4\tau\omega_i(1 + \tau\omega_i)\left[H + 4\xi^2(1 + \tau\omega_i)\right]}{4\ ^2(1 - \tau^2\omega_i^{\,2})}\, e^{a^2/2\sigma^2} \tag{124}$$

Equation (124) may be solved graphically by plotting the left and right hand sides of the equation against $\frac{a}{\sigma}$. Where there are intersections of the two curves in the plot, equation (124) is satisfied and the fact that the conditions for the jump phenomenon exist have been established.

If, in connection with the same system, the input is now changed so that it is equivalent to white noise passed through a second order lag filter, the transfer function from Figure 20 is

$$H(\omega) = \frac{\dfrac{1}{j\omega C}}{R + j\omega L + \dfrac{1}{j\omega C}}$$

$$= \frac{1}{LC(j\omega)^2 + RC(j\omega) + 1}$$

$$= \frac{\dfrac{1}{LC}}{(j\omega)^2 + \dfrac{R}{L}(j\omega) + \dfrac{1}{LC}}$$

$$= \frac{\ _n^{\,2}}{(j\omega)^2 + 2\xi_n\omega_n(j\omega) + \omega_n^{\,2}}$$

$$= \frac{1}{(\frac{j\omega}{\omega_n})^2 + 2\xi_n(\frac{j\omega}{\omega_n}) + 1} \qquad (125)$$

where

$$\omega_n^2 = \frac{1}{LC}$$

$$\xi_n = R\sqrt{\frac{C}{L}}$$

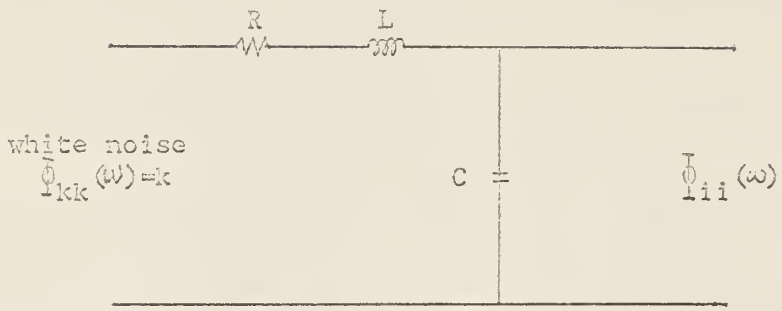


Figure 20. Input output relation of second order lag filter.

Thus,

$$\Phi_{ii}(\omega) = |H(\omega)|^2 \, \Phi_{kk}(\omega)$$

$$= k \left| \frac{1}{(\frac{j\omega}{\omega_n})^2 + 2\xi_n(\frac{j\omega}{\omega_n}) + 1} \right|^2 \qquad (126)$$

The power spectral density of the error, e(t), is

$$\Phi_{ee}(\omega) = \left| \frac{E(\omega)}{I(\omega)} \right|^2 \Phi_{ii}(\omega)$$

$$= k \left| \frac{(j\omega)(j\omega + \frac{1}{\tau})}{(j\omega)^2 + \frac{j\omega}{\tau} + \frac{HK}{\tau}} \right|^2 \cdot \left| \frac{1}{(\frac{j\omega}{\omega_n})^2 + 2\xi_n (\frac{j\omega}{\omega_n}) + 1} \right|^2$$

$$= k\omega_n^2 \left| \frac{(j\omega)^4 - \frac{1}{\tau^2}(j\omega)^2}{(j\omega)^4 + (\frac{1}{\tau} + 2\xi_n\omega_n)(j\omega)^3 + (\frac{HK}{\tau} + \frac{2\xi_n\omega_n}{\tau} + \omega_n^2)(j\omega)^2 + \frac{1}{\tau}(2\xi_n HK\omega_n + \omega_n^2)(j\omega) + \frac{HK}{\tau}\omega_n^2} \right|^2 \tag{127}$$

The mean square value of the error signal is then

$$\overline{e^2(t)} = \Phi_{ff}(0) = \frac{1}{2} \int_{-\infty}^{\infty} \Phi_{ee}(\omega)d\omega$$

$$= \frac{k\omega_i^2}{2} \cdot \frac{2\pi}{2\pi j} \int_{-\infty}^{\infty} \frac{b_o x^6 + b_1 x^4 + b_2 x^2 + b_3}{\left| a_o x^4 + a_1 x^3 + a_2 x^2 + a_3 x + a_4 \right|^2} dx$$

$$= (k\omega_n^2) I_4 \tag{128}$$

where

$$a_o = 1 \qquad b_o = 0$$

$$a_1 = \frac{1}{\tau} + 2\xi_n\omega_n \qquad b_1 = 1$$

$$a_2 = \frac{HK + 2\xi_n\omega_n}{\tau} + \omega_n^2 \qquad b_2 = -\frac{1}{\tau^2}$$

$$a_3 = \frac{2\xi_n HK\omega_n + \omega_n^2}{\tau} \qquad b_3 = 0$$

$$a_4 = \frac{HK\omega_1^{\,2}}{\tau} \qquad\qquad x = j\omega$$

From table in appendix:

$$I_4 = \frac{b_o(-a_1a_4 + a_2a_3) - a_oa_3b_1 + a_oa_1b_2 + \frac{a_ob_3}{a_4}(a_oa_3 - a_1a_2)}{2a_o(a_oa_3^{\,2} + a_1^{\,2}a_4 - a_1a_2a_3)} \tag{129}$$

substituting the appropriate values for the constants, there results

$$I_4 = \frac{-\frac{2}{\tau}(\xi_n HK\omega_n + \omega_n^{\,2}) - \frac{1}{\tau^2}(\frac{1}{\tau} + 2\xi_n\omega_n)}{2\left[\frac{1}{\tau^2}(2\xi_n HK\omega_n + \omega_n^{\,2})^2 + (\frac{1}{\tau} + 2\xi_n\omega_n)^2 \frac{HK\omega_n^{\,2}}{\tau}\right.}$$

$$\left. - (\frac{1}{\tau} + 2\xi_n\omega_n)(\frac{HK + 2\xi_n\omega_n}{\tau} + \omega_n^{\,2})(\frac{2\xi_n HK\omega_n + \omega_n^{\,2}}{\tau})\right]$$

$$= \frac{\frac{2}{\tau}(\xi_n HK\omega_n + \omega_n^{\,2}) + \frac{1}{\tau^2}(\frac{1}{\tau} + 2\xi_n\omega_n)}{\left[\frac{1}{\tau^3}(2\xi_n H^2K^2\omega_n + 4\xi_n^{\,2}HK\omega_n^{\,2} + 2\xi_n\omega_n^{\,3})\right.}$$

$$\left. + \frac{1}{\tau^2}(8\xi_n^{\,3}HK\omega_n^{\,3} - 4\xi_n HK\omega_n^{\,3} + 4\xi_n^{\,2}\omega_n^{\,4}) + \frac{2}{\tau}\xi_n\omega_n^{\,5}\right]$$

$$= \frac{(1 + \omega_n^{\,2}\tau^2) + 2\xi_n\omega_n\tau(HK\tau + 1)}{2\omega_n\xi_n\left[(\omega_n^{\,2}\tau - HK)^2 + (2\xi_n\omega_n\tau + 1)(2HK\xi_n\omega_n + \omega_n^{\,2})\right]} \tag{130}$$

The mean square value of the error is, then,

$$\overline{e^2} = \frac{k\omega_n \pi}{2\xi_n} \cdot \frac{(1 + \omega_n^2 \tau^2) + 2\xi_n \omega_n \tau (HK\tau + 1)}{(\omega_n^2 \tau - HK)^2 + (2\xi_n \omega_n \tau + 1)(2HK\xi_n + \omega_n^2)} \qquad (131)$$

The mean square value of the input is

$$\overline{i^2} = \frac{1}{2} \int_{-\infty}^{\infty} \Phi_{ii}(\omega) d\omega$$

$$= \frac{k}{2} \int_{-\infty}^{\infty} \frac{1}{\left| \left(\frac{j\omega}{\omega_n}\right)^2 + 2\xi_n \left(\frac{j\omega}{\omega_n} \right) + 1 \right|^2} d\omega$$

$$= \frac{k\omega_n}{2} \int_{-\infty}^{\infty} \frac{1}{\left| - \left(\frac{\omega}{\omega_n}\right)^2 + 2j\xi_n \left(\frac{\omega}{\omega_n}\right) + 1 \right|^2} d\left(\frac{\omega}{\omega_n} \right)$$

$$= \frac{k\omega_n}{2} \int_{-\infty}^{\infty} \frac{b_o x^2 + b_1}{\left| a_o x^2 + a_1 x + a_2 \right|^2} dx$$

$$= (j\pi k\omega_n) \, I_2 \qquad (132)$$

where

$a_o = -1$ $\qquad$ $b_0 = 0$

$a_1 = 2j\xi_n$ $\qquad$ $b_1 = 1$

$a_2 = 1$ $\qquad$ $x = \frac{\omega}{\omega_n}$

From table in appendix:

$$I_2 = \frac{-b_o + \dfrac{a_o b_1}{a_2}}{2a_o a_1} \tag{133}$$

substituting the appropriate values for the constants, there is obtained

$$I_2 = \frac{+ 1}{+4j\xi_n} \tag{134}$$

therefore

$$\overline{i^2} = j\pi k\omega_n \cdot \frac{1}{4j\xi_n}$$

$$= \frac{\pi k\omega_n}{4\xi_n} \tag{135}$$

Upon substituting equation (135) into (131) there is obtained

$$\overline{e^2} = 2\overline{i^2} \left[\frac{(1 + \omega_n^2\tau^2) + 2\xi_n\omega_n (HK\tau + 1)}{(\omega_n^2\tau - HK)^2 + (2\xi_n\omega_n\tau + 1)(2HK\xi_n + \omega_n^2)} \right] \tag{136}$$

The mean square value of the input to the limiter is related to the mean square value of the system error signal by the expression

$$\sigma^2 = K^2\overline{e^2}$$

$$= 2K^2\overline{i^2} \left[\frac{(1 + \omega_n^2\tau^2) + 2\xi_n\omega_n (HK\tau + 1)}{(\omega_n^2\tau - HK)^2 + (2\xi_n\omega_n\tau + 1)(2HK\xi_n + \omega_n^2)} \right] \tag{137}$$

or

$$2K^2\overline{i^2} = \frac{(\omega_n^2\tau - HK)^2 + (2\xi_n\omega_n\tau + 1)(2HK\xi_n + \omega_n^2)}{(1 + \tau^2\omega_n^2) + 2\xi_n\omega_n\tau(1 + \tau HK)}\,\sigma^2 \qquad (138)$$

Equation (138) is plotted on Figure 21, in terms of the relationship between σ^2 and $\overline{i^2}$. Because the values of H in equation (138) itself is a function of σ, one can find the corresponding values of $\overline{i^2}$ by simply assuming some values of σ^2. The following table is calculated by putting a=1, (a is the value that saturating amplifier begin to saturate), ξ=0.1, ξ_n=0.05.

It may be readily observed from Figure 22, in certain case, there are two or more values of σ^2 for some values of $\overline{i^2}$. In other words, there are two or more values of e^2 corresponds to a certain value of $\overline{i^2}$. This shows the condition for the jump phenomenon with random input.

Table 1. The corresponding values of Figure 22 according to equation (138).

σ	σ^2	H	$\omega_n \tau$	$\overline{2i^2}$	$\omega_n \tau$	$\overline{2i^2}$	$\omega_n \tau$	$\overline{2i^2}$
0	0	1	3.5	0	4.0	0	3.0	0
0.500	0.25	0.957	3.5	0.00258	4.0	0.00144	3.0	0.00560
0.707	0.50	0.843	3.5	0.00324	4.0	0.00186	3.0	0.00804
0.866	0.75	0.760	3.5	0.00320	4.0	0.00202	3.0	0.00892
1.000	1.00	0.683	3.5	0.00270	4.0	0.00218	3.0	0.00883
1.118	1.25	0.628	3.5	0.00213	4.0	0.00262	3.0	0.00840
1.224	1.50	0.583	3.5	0.00205	4.0	0.00341	3.0	0.00802
1.323	1.75	0.550	3.5	0.00197	4.0	0.00428	3.0	0.00787
1.414	2.00	0.521	3.5	0.00194	4.0	0.00523	3.0	0.00767
1.500	2.25	0.494	3.5	0.00221	4.0	0.00694	3.0	0.00730
1.581	2.50	0.472	3.5	0.00271	4.0	0.00844	3.0	0.00714
1.658	2.75	0.453	3.5	0.00326	4.0	0.01002	3.0	0.00707
1.732	3.00	0.436	3.5	0.00386	4.0	0.01190	3.0	0.00708

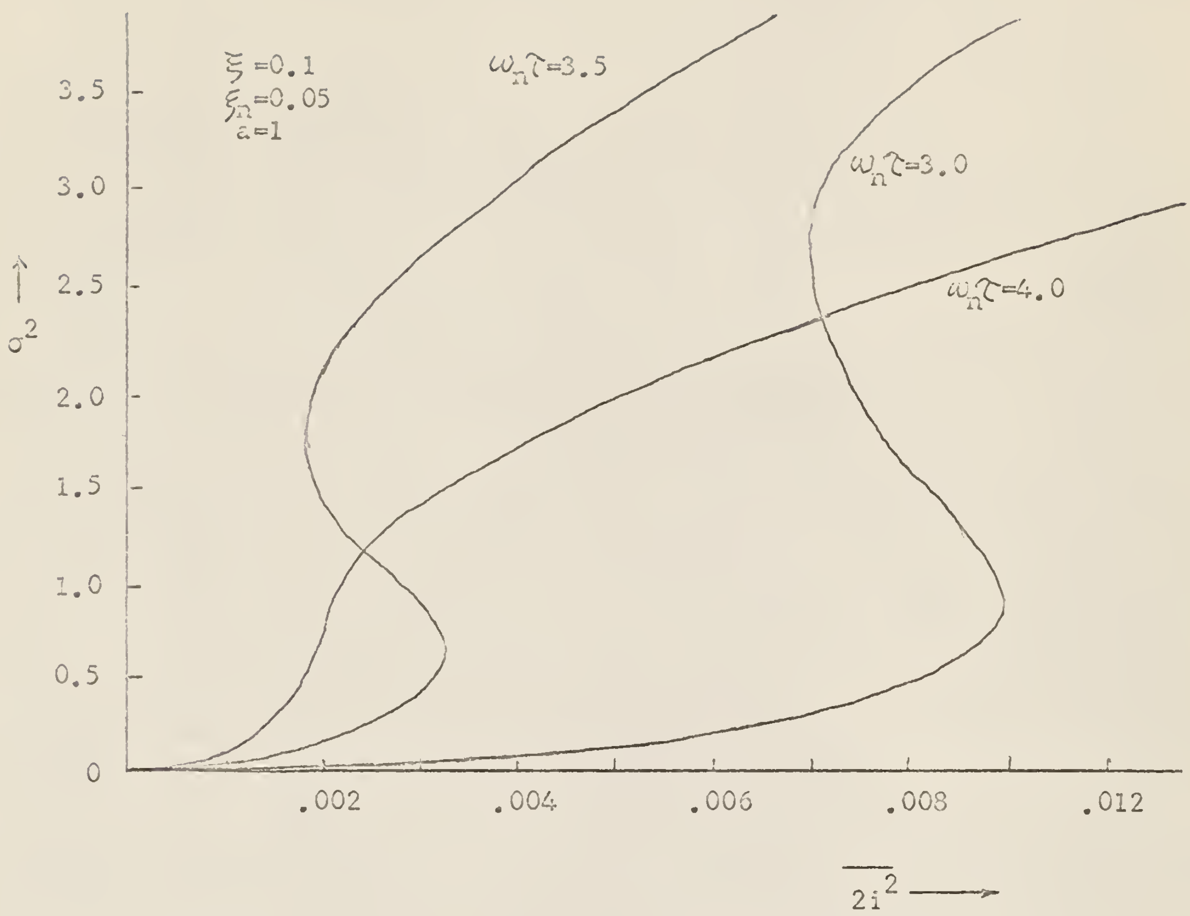


Figure 21. Performance of the limiting amplifier servomechanism exhibiting the jump phenomenon.

CONCLUSION

The methods of analysis in this report are restricted to the simple nonlinear devices in which the outputs are functions of the instantaneous values of the inputs, i.e., the devices have no energy storing properties.

The inputs to position servomechanism with a saturating amplifier that exhibits the jump phenomenon are discussed for the case when a white noise passes the first and the second order lag-filter. When a white noise passes through higher order lag-filter the analysis becomes rather complicated. It is hoped that the writer may have a chance to deal with this case in the near future.

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APPENDIX

This is adapted from the table in H. M. James, N. B. Nichols, R. S. Phillips, Theory of Servomechanisms, Mc Graw-Hill Book Co., New York, 1947, pp. 369-370.

If there is integral of the form

$$I_n = \frac{1}{2\pi j} \int_{-\infty}^{\infty} \frac{g_n(x)}{h_n(x)\ h_n(-x)}\ dx$$

where

$$h_n(x) = a_o x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n$$

$$g_n(x) = b_o x^{2n-2} + b_1 x^{2n-4} + \cdots + b_{n-2} x^2 + b_{n-1}$$

and roots of $h_n(x)$ all lie in the upper half plane. The table lists the integrals I_n for values of n from 1 to 5.

$$I_1 = \frac{b_o}{2a_o a_1}$$

$$I_2 = \frac{-b_o + \dfrac{a_o b_1}{a_2}}{2a_o a_1}$$

$$I_3 = \frac{-a_2 b_o + a_o b_1 - \dfrac{a_o a_1 b_2}{a_3}}{2a_o (a_o a_3 - a_1 a_2)}$$

$$I_4 = \frac{b_o(-a_1a_4 + a_2a_3) - a_oa_3b_1 + a_oa_1b_2 + \frac{a_ob_3}{a_4}(a_oa_3 - a_1a_2)}{2a_o(a_oa_3^2 + a_1^2a_4 - a_1a_2a_3)}$$

$$I_5 = \frac{b_o(-a_oa_4a_5 + a_1a_4^2 + a_2^2a_5 - a_2a_3a_4) + a_ob_1(-a_2a_5 + a_3a_4)}{a_o^2a_5^2 - 2a_oa_1a_4a_5 - a_oa_2a_3a_5 + a_oa_3^2a_4} \cdots$$

$$\frac{+ a_ob_2(a_oa_5 - a_1a_4) + a_ob_3(-a_oa_3 + a_1a_2) + \frac{a_ob_4}{a_5}}{+ a_1^2a_4^2 + a_1a_2^2a_5 - a_1a_2a_3a_4} \cdots$$

$$\frac{(-a_oa_1a_5 + a_oa_3^2 + a_1^2a_4 - a_1a_2a_3)}{}$$

THE ANALYSIS OF NONLINEAR SYSTEMS
SUBJECT TO RANDOM INPUTS

by

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There are a number of methods for analyzing the effect of random inputs of the dynamic error of control systems that contain nonlinear elements. One method involves replacement of the nonlinear element in the system by an equivalent device whose describing function characteristics are functions of the rms signals at the input of nonlinear element. Two methods of developing statistical describing functions are given in this report. One of them uses an analytical method to minimize the system error signals. The other requires the determination of the output probability distribution of the nonlinear device and defines the ratio of the rms value of the output to input distribution as the statistical describing function. Both of these methods are applied to simple nonlinear devices in which the output is a function of the instantaneous value of the input.

A rather interesting example of jump phenomenon occurs when the position servomechanism with a saturating amplifier is subjected to a particular input signal. This phenomenon has also been discussed.

A brief discussion of some fundamental statistical techniques is also given in order to provide the supporting background for the development of the analysis.